On durable goods markets with entry and adverse selection

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Abstract. We investigate the nature of trading and sorting induced by the dynamic price mechanism in a competitive durable good market with adverse selection and exogenous entry of traders over time. The model is a dynamic version of Akerlof (1970). Identical cohorts of durable goods, whose quality is known only to potential sellers, enter the market over time. We show that there exists a cyclical equilibrium where all goods are traded within a finite number of periods after entry. Market failure is reflected in the length of waiting time before trade. The model also provides an explanation of market fluctuations. JEL classification: D82

A propos des marchés de biens durables quand il y a entrée de nouveaux commerçants et sélection adverse. Les auteurs analysent la nature du commerce et du triage engendrés par le mécanisme dynamique des prix dans un marché concurrentiel de biens durables quand il y a sélection adverse et entrée exogène de nouveaux commerçants dans le temps. Ce modèle est une version dynamique du modèle d' Akerlof (1970). Des cohortes identiques de biens durables, dont la qualité est connue seulement des vendeurs potentiels, arrivent sur le marché dans le temps. Il semble qu'il y ait plus de commerce actif que ce qui est prévu par un modèle statique. En particulier, on montre qu'il existe un équilibre cyclique où tous les biens sont transigés à l'intérieur d'un nombre fini de périodes après leur arrivage et que, à chaque phase du cycle, l'éventail de qualité des biens transigés s'accroît. Les commerçants qui transigent des produits de plus haute qualité attendent plus longtemps et l'imperfection du marché se traduit par la longueur de temps d'attente avant la transaction. Le modèle fournit aussi une explication des fluctuations du marché.

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1. Introduction

A fundamental problem of resource allocation in competitive markets is the difficulty of conducting potentially gainful trade when some traders have private information about certain characteristics that affect the utility from trading. The pioneering paper by Akerlof (1970) illustrated this in the context of a static market for used goods where the quality of goods is known only to potential sellers and as a result, higher-quality goods cannot be traded (see also Wilson 1979). This *adverse selection* (or 'lemons') problem occurs in a wide range of markets where traders are asymmetrically informed and have correlated valuations such as markets for insurance (automobile and health insurance);[1] labour (see Greenwald 1986); financial assets (credit) (see, e.g., Spulber 1999, chap. 8); and even the market for thoroughbred yearlings (horses) (see, e.g., Chezum and Wimmer 1997).

A large number of these markets involve trading in durable goods or assets. In such markets, traders may wait before they trade thereby creating variations in the distribution of types of traders over time. This introduces the possibility that the dynamic price system can induce sorting by giving incentives to existing agents with different private information to trade in different time periods. However, new traders may enter the market over time. Entry of new agents with private information can increase the extent of incomplete information in the market. Our main goal in this paper is to understand the extent to which the competitive price system can, by itself, act as a sorting mechanism despite the entry of new traders.

Empirically, pure adverse selection effects have not been frequently 'observed', and markets characterized by asymmetric information do appear to exhibit fairly active trading (see, e.g., Bond 1982, 1984). One of the main reasons behind this is the existence of institutional arrangements and technologies that generate incentives for traders to take actions that modify the information structure in the market. Informed agents may *signal* their private information through observable actions and uninformed agents may develop mechanisms to *screen* or distinguish informed agents with differing information.[2] Our paper shows that even when there are no other institutions or technologies available for agents to undertake screening or signalling activities, and even when new traders with private information enter the market over time, the dynamic price mechanism can ensure a high volume of trading – certainly higher than that predicted by static models.[3]

1 Dionne and Doherty (1994) give a literature survey on insurance and adverse selection.

2 For example, in markets for used cars, the introduction of systems of guarantees and certification intermediaries (Biglaiser and Friedman (1999) and Lizzeri (1999)) can reduce the extent of the 'lemons' problem.

3 For an analysis of constrained efficiency of static competitive markets for 'lemons,' see Bigelow (1990) and Gale (1992, 1996).

Recently, a growing literature on used durable goods markets has brought out the role of time and the interaction between 'primary' and 'secondary' markets in resolving the 'lemons' problem. In particular, when one takes into account how the market for new goods influences the characteristics of agents who choose to trade in used goods markets, one can explain why used goods markets appear to be much more active than what is predicted by the Akerlof model. Thus, Hendel and Lizzeri (1999) have argued that used car sellers are likely to be higher-valuation buyers who are eager to buy new cars – they are more sensitive to quality and have a high opportunity cost of holding on to a lemon – and this eliminates much of the adverse selection problem in the resale market (see also Anderson and Ginsburgh 1994). Similarly, Guha and Waldman (1997), Hendel and Lizzeri (2002), and Waldman (1999) have emphasized the role of leasing (rather than selling) in the market for new durable goods in reducing the informational asymmetry in the used goods market.

Our explanation of why markets with asymmetric information can exhibit fairly active trading is qualitatively different and complements the literature mentioned above. Instead of focusing on inter-market interaction, our analysis is confined to the properties of the dynamic price mechanism *within* a given market. This also means that we specify the sets of traders and their characteristics exogenously and study the functioning of the competitive market mechanism for any such given specification. In this context, it is important to point out that used goods markets are not the only markets subject to the 'lemons problem.' In other markets, such as those for long-term insurance, labour contracts, or financial assets, interactions between 'primary' and 'secondary' markets are less important.

Our central concern in this paper is the same as that in Akerlof (1970) and Wilson (1979). The framework adopted is a natural extension of the static model analysed in these papers to a dynamic setting with entry. In particular, we consider a competitive market for a durable good where potential sellers are privately informed about the quality of the goods they are endowed with. In each period, a cohort of sellers of equal size and with an identical distribution of quality enters the market. The distribution of quality is uniform within each cohort. Buyers are identical, have unit demand, and for any given quality, a buyer's willingness to pay exceeds the reservation price of a seller for that quality. As buyers do not know the quality, their willingness to pay is their expected valuation of goods offered for sale in that period.[4] For simplicity, we assume that there are more buyers than sellers in each period, so that in equilibrium, prices equal the expected valuation of buyers and all buyers earn zero surplus. Once traded, goods are not resold in the same market.[5]

4 Our analysis bears some resemblance to that by Sobel (1991) of a durable goods monopoly where new cohorts of consumers enter the market over time. Unlike our framework, there is no correlation between the valuations of buyers and sellers in his model.

5 For an analysis of dynamic Walrasian markets with complete information and entry of traders over time, see Wooders (1998).

The static version of this model is simply the Akerlof-Wilson model; as is well known, the unique equilibrium outcome in this case is that only a certain low range of quality is traded in the market; the buyers' valuation of the average quality in this range (the price) exactly matches the seller's reservation price for the highest quality traded. The infinitely repeated version of this outcome, where the same qualities are traded in every period (while higher-quality goods remain unsold forever) is also an equilibrium in our dynamic model. In fact, it is the unique stationary equilibrium (and the only one where prices and average quality traded are monotonic over time).

There are, however, other equilibria with more interesting properties – where prices and average quality traded fluctuate over time. In all such equilibria, the range of quality that is eventually traded in the market exceeds that in the stationary (static) outcome. Within each cohort of entrants, the owner of a good with lower quality trades earlier, owners of higher-quality goods wait longer. In the static Akerlof-Wilson model, the adverse selection problem manifests itself in the fact that relatively high-quality goods cannot be traded despite the potential gains from trade. In the dynamic market for durable goods, the *lemons problem* may manifest itself in the fact that sellers with relatively high-quality goods need to wait longer in order to trade.[6]

We prove the existence of a cyclical equilibrium where *every* potential seller entering the market trades within a certain finite number of periods after entering the market. On the equilibrium path, prices and the range of quality traded increase until all potentially tradable goods (including those of the highest possible quality) are traded in a particular period. The process repeats from the next period onwards. In this equilibrium, market inefficiency arises *only* because sellers with goods of higher quality must wait. We provide examples of other cyclical equilibria where prices and the range of quality traded are non-monotonic within each cycle and also those where the top-quality goods are never traded, though the range of qualities traded is better than in the static outcome.[7]

There are three important factors determining market dynamics in all the non-stationary equilibria of our model. First, once a certain range of quality is traded, only sellers of higher-quality goods are left in the market, improving

6 There are situations in which the fact that a seller has waited for a long time indicates low rather than high quality. This is true, for example, when buyers can inspect quality: (high) valuation buyers are more likely to inspect and select relatively high-quality houses, leaving unsold goods of relatively low quality for later periods (Taylor 1999; Vettas 1997). As stated earlier, our model is designed to understand the nature of the lemons problem, so we do not allow for any technology that can directly modify the information structure.

7 A follow-up paper by Janssen and Karamychev (2002) extends the result about greater volume of trading in a dynamic setting to a more general class of probability distributions. Confining attention to the uniform distribution case allows us to establish a very specific type of dynamic equilibrium (which is also simple and intuitive) viz., one where the range of quality expands over time until all goods in the market are sold. We are also able to provide a much tighter qualitative characterization of equilibria.

the quality distribution of goods in the future. Second, the entry of a new cohort of potential sellers with goods of all possible quality dilutes the average quality of potentially tradable goods, since buyers cannot distinguish them from higher-quality sellers who entered in the past. Finally, as time progresses and the stock of unsold goods accumulates, the new cohort of traders entering the market in any period becomes increasingly less significant in determining the distribution of quality of tradable goods.

If there is no entry of sellers after the initial period (or equivalently, if buyers can distinguish the period of entry of sellers in the market), then only the first factor is relevant. In fact, the problem can then be analysed in a simpler model, where there is only a given set of sellers at time zero and there is no entry over time. This model has been analysed in an earlier paper (Janssen and Roy 2002), and it has been shown that in *every* equilibrium all goods are traded in finite time.[8] A dynamic auction game with similar features is analysed by Vincent (1990).[9]

Our results can also help us to understand some issues explored in the literature on financial markets. In such markets, the adverse selection problem arising from asymmetric information (about assets and returns from potential investment) between managers of firms and potential investors provides a key explanation for why firms prefer not to issue equities and why announcement of equities leads to significant price drops (Myers and Majluf 1984). Empirical studies have established intertemporal *fluctuation*s in volume of equities and the extent to which equity issues have a downward impact on their prices (see, e.g., Korajczyk, Lucas, and McDonald 1991). These fluctuations have been explained in terms of exogenous changes in the extent of asymmetric information. In our paper, though it is not directly cast in the framework of financial markets, we indicate that even if there are no exogenous factors that change over time, changes in the extent of adverse selection problem can emerge *endogenously*. Thus, firms with better private valuations may delay issuing equities because they expect similar firms in their cohort to do so, and the investors infer that which, in turn, reduces the price impact when these firms actually issue their equities.[10]

8 Indeed, the present paper can be considered a companion piece, where the effects of entry on the dynamic market equilibria are analysed.

9 Models of sequential bargaining with one-sided incomplete information also highlight the use of intertemporal price discrimination as a sorting device. Vincent (1989) and Evans (1989) analyse the case of correlated valuation. In fact, the main idea behind the reasoning here can be traced back to the analysis of a static competitive market for lemons by Wilson (1980), where the role of price dispersion in sorting of sellers is emphasized.

10 Lucas and McDonald (1990) analyse a stationary dynamic model where changes in the information structure and delay in issue emerges endogenously. However, it is *assumed* that private information about assets of firms in each period is publicly observed in the next period. In our model, private information is never directly observed.

Finally, all of the non-stationary equilibria of our model are necessarily non-monotonic in prices and average quality traded. The non-monotone dynamics in market variables as well as the multiplicity of equilibria and the resultant coordination problem are manifestations of the *lemons problem* in dynamic markets.[11]

The paper is organized as follows. In section 2 we set out the model and the equilibrium concept. The main results of the paper relating to existence and properties of different types of equilibria are outlined in section 3. section 4 concludes. Proofs are contained in the appendix.

2. Preliminaries

Consider a Walrasian market for a perfectly durable good[12] whose quality, denoted by θ, varies between $\underline{\theta}$ and $\overline{\theta}$. Time is discrete and is indexed by $t = 1, 2, \ldots \infty$. All agents discount their future return from trading using a common discount factor δ, $0 \leq \delta < 1$. There is an unbounded continuum of infinitely lived potential sellers; formally, we let each real number larger than 1 be a potential seller. In each period a cohort of sellers enters the market; all cohorts are of equal size (normalized to be equal to 1). The set of sellers entering the market in period t is denoted by I_t. We will assume that each I_t is equal to the segment $(t, t+1)$. Each seller is endowed with one unit of the durable good at the point of entry into the market; seller i knows the quality $\theta(i)$ of the good he is endowed with and derives flow utility from ownership of the good till he sells it. Seller i's valuation (reservation price) of the good is his infinite horizon discounted sum of utility derived from ownership of the good (if he does not sell it), and we assume that it is exactly equal to $\theta(i)$. This implies that the per period flow utility from owning the good is $(1-\delta)\theta(i)$. For each t, the cohort of entering sellers (i.e., sellers in I_t) own goods whose quality is *uniformly* distributed over $[\underline{\theta}, \overline{\theta}], 0 \leq \underline{\theta} < \overline{\theta} < +\infty$.

In each period, a continuum of (potential) buyers enter the market whose (Lebesgue) measure is greater than 1. This ensures that in every period the sellers are on the short side of the market. All buyers are identical and have unit demand. A buyer's valuation of a unit of the good with quality θ is equal to $v\theta$, where $v > 1$. Thus, for any specific quality, a buyer's valuation exceeds the seller's. Buyers know the ex ante distribution of quality, but do not know the quality of the good offered by any particular seller. Once trade occurs, the buyer leaves the market with the good she has bought, that is, there is no scope

11 In the static version of this model, Wilson (1979, 1980) pointed out the possibility of multiple equilibria. The multiplicity of equilibria in his model is due to the nature of the initial distribution of quality. For the case of uniformly distributed quality, there is a unique non-trivial equilibrium in the static model. The source of multiplicity of equilibria in our model lies in its dynamic aspects (in combination with the asymmetry of information).

12 In section 4, we briefly indicate how the analysis may be adapted to some form of depreciation.

for reselling.[13] In order to ensure that there is an adverse selection problem in the static version of this model, we will assume $vE(\theta) < \overline{\theta}$, where $E(\theta)$ is the unconditional expected quality, which is equal to $(\underline{\theta} + \overline{\theta})/2$.

Given a sequence of anticipated prices $\mathbf{p} = \{p_t\}_{1,2\ldots\infty}$, each seller chooses whether or not to sell and if he chooses to sell, the time period in which to sell. A seller entering the market in period t with quality θ can always earn a utility level of θ by not selling. If he sells in period $t+j$, his *net* benefit of selling is

$$\sum\nolimits_{i=0,\ldots,j-1}[(1-\delta)\theta]\delta^i + \delta^j p_{t+j} - \theta = \delta^j(p_{t+j} - \theta),$$

that is, he gets the flow utility in the first j periods, then he receives the price in period $t+j$ when he gives up ownership. The set of time periods in which a potential seller entering in period t with quality θ (given prices $\mathbf{p}$) finds it optimal to sell is denoted by $T(\theta,\mathbf{p},t)$. Let us denote by N^* the extended set of positive natural numbers including the point $+\infty$. Then,

$$T(\theta,\mathbf{p},t) = \{\tau \geq t : \tau \in N^* \text{ and } \delta^{\tau-t}(\theta - p_\tau) \geq \delta^{t'-t}(\theta - p_{t'}) \text{ forall } t' \geq t\}.$$

Each potential seller $i \in I_t$ chooses a particular time period $\tau(i,\mathbf{p},t) \in T(\theta(i),\mathbf{p},t)$ in which to sell. Choosing to sell in period $+\infty$ is equivalent to not selling at all. This, in turn, generates a certain distribution of quality among goods offered for sale in each time period τ. Given $\mathbf{p}$, the average quality of goods offered for sale in period t is $E\{\theta(i) \mid i \in I_k,\ k \leq t,\ \tau\ (i,\mathbf{p},k) = t\}$.

On the equilibrium path, buyers' expected quality in periods where trade occurs equals the average quality sold. As all potential buyers are identical, we will assume that belief about quality is symmetric. A dynamic equilibrium is one where all agents maximize their objectives, expectations are fulfilled, and markets clear every period.

DEFINITION. *A dynamic equilibrium is given by a price sequence* $\mathbf{p} = \{p_t\}_{t=1,2\ldots\infty}$, *a set of selling decisions* $\tau(i,\mathbf{p},t)$, $i \in I_t$, $t = 1,2,\ldots$, *and a sequence* $\{E_t(\mathbf{p})\}_{t=1,2\ldots\infty}$, *where* $E_t(\mathbf{p})$ *is the (symmetric) expectation of quality in period t held in common by all potential buyers in the market in period t, such that*:

(*i*) ***Sellers maximize***: $\tau(i,\boldsymbol{p},t) \in T(\theta(i),\boldsymbol{p},t)$; *that is, it is optimal for seller i with quality* $\theta(i)$ *to sell in period* $\tau(i,\boldsymbol{p})$;

(*ii*) ***Buyers maximize and markets clear***: *If* $\mu\{i \mid i \in I_k,\ k \leq t,\ \tau(i,\boldsymbol{p},k) = t\} > 0$, *then* $p_t = vE_t(\boldsymbol{p})$, *where* $\mu\{\cdot \mid \cdot\}$ *is the measure of sellers preferring to sell in period t; that is, if strictly positive measure of trade occurs, then buyers must be indifferent between buying and not buying, so that the market clears. If*

13 Allowing for reselling of goods by buyers introduces elements of a rental market into the framework and, to some extent, diverts attention from the basic issues addressed in this paper. If the number of transactions a particular commodity has undergone is publicly observable (e.g., when 'first-hand,' 'second-hand,' 'third-hand' goods are distinguished), then the market where a buyer resells the good is separate from the one where she initially buys.

$\mu\{i \mid i \in I_k,\ k \leq t,\ \tau(i,\mathbf{p},k) = t\} = 0$, *then* $p_t \geq vE_t(\mathbf{p})$; *that is, if no trade occurs, then it must be optimal for buyers to not buy in that period.*[14]

(*iii*) ***Expectations are fulfilled***: *The expectations of quality in periods of positive trading must exactly equal the average quality of goods sold in that period; that is,* $E_t(\mathbf{p}) = E\{\theta(i) \mid i \in I_k,\ k \leq t,\ \tau(i,\mathbf{p},k) = t\}$, if $\mu\{i \mid i \in I_k,\ k \leq t,\ \tau(i,\mathbf{p},k) = t\} > 0$.

(*iv*) ***Expected quality under no trade***: *Even if no trade occurs in a period, buyers' expected quality is at least as large as the lowest possible quality; that is,*

$$E_t(\mathbf{p}) \geq \underline{\theta} \text{ for all } t. \tag{1}$$

While conditions (*i*)–(*iii*) are standard, condition (iv) simply imposes a weak constraint on the equilibrium belief about quality held by buyers (i.e., their willingness to pay) in periods in which the measure of goods traded is zero.

3. Equilibria

In this section, we discuss the nature of market equilibria and establish the existence of certain types of equilibria. The results below provide insight into the way in which asymmetric information and the possibility of waiting affect the functioning of the market mechanism and the dynamics of prices and characteristics of trades over time.

First, we state a fundamental property of *all* equilibria: if a good of a certain quality is traded in a particular period, then all goods of *lower* quality held by sellers that are currently in the market (and have not yet traded) must be traded in that period. To see this, note that a seller whose good is of quality x finds it optimal to sell in period t only if

$$p_t - x \geq \delta^{\tau-t}(p_\tau - x) \text{ for all } \tau > t.$$

This, in turn, implies that for any seller with good of quality $y < x$

$$p_t - y > \delta^{\tau-t}(p_\tau - y).$$

The incentive to wait (the difference in discounted surplus obtained by not selling and waiting till a particular period) for a higher-quality seller always exceeds that for a seller with lower quality. Thus, in every period t in which trade occurs, the set of qualities traded is an interval $[\underline{\theta},\theta_t]$, where θ_t is the *marginal quality traded in period t*; all sellers who have entered the market by that period, have not traded in an earlier period and whose goods are of quality lying in this interval trade in period t.

14 Remember that the measure of buyers exceeds the total measure of sellers, so that if the price falls below $vE_t(\mathbf{p})$, there would be excess demand in the market.

An important property of all equilibria follows from the above discussion: *within the cohort of sellers entering the market in a particular period, a seller of higher quality never sells earlier than a seller of lower quality*. In particular, if the equilibrium is non-stationary such that within the same cohort of sellers, some sell earlier and others later, then it is the higher-quality sellers who wait longer in order to sell. Aside from the fact that certain high-quality goods may never be traded in some of the equilibria, the cost of waiting for high-quality sellers is a significant component of market failure.

PROPOSITION 1. *In any period t in which trade occurs, the set of qualities traded is an interval* $[\underline{\theta},\theta_t]$. *Within the cohort of sellers entering the market in any particular period, sellers of higher quality never trade earlier than sellers of lower quality.*

Recall that we have assumed that

$$vE(\theta) < \overline{\theta}.^{15} \tag{2}$$

Under this assumption, in the one-period version of our model (the Akerlof-Wilson framework), the unique non-trivial equilibrium is that only goods of quality lying in the interval $[\underline{\theta},\theta_s]$ may be traded, where θ_s is uniquely determined by

$$v\frac{\underline{\theta} + \theta_s}{2} = \theta_s. \tag{3}$$

We shall refer to θ_s as the '*static quality*' – the understanding being that this is the highest *marginal* quality sold in the single-period version of our model.

It is easy to check that even in our dynamic model, an outcome where entering sellers with quality lying in $[\underline{\theta},\theta_s]$ trade at price θ_s in every period, is an equilibrium. Prices are constant over time, so no entering seller with quality in the interval $[\underline{\theta},\theta_s)$ has an incentive to wait and sell later. The seller with quality θ_s earns exactly zero surplus and is indifferent between selling, not selling, and selling later; sellers with quality higher than θ_s find that the market price is always below their reservation price. In fact, this is the unique stationary equilibrium of our model. To see this, observe that in any stationary equilibrium, the interval of quality traded must be identical over time, and the requirement that sellers with quality outside this interval should not wish to trade, together imply that the seller of the marginal quality traded must earn zero surplus in every period. It follows that the marginal quality must be θ_s in period 1 and hence in every period.

15 Note that this is equivalent to $v < 2\overline{\theta}/(\overline{\theta} + \underline{\theta})$, which is always smaller than 2.

PROPOSITION 2. *The infinitely repeated version of the unique non-trivial equilibrium of the one-period version of our model is also the unique stationary equilibrium of our model. In this equilibrium, in every period entering sellers with quality lying in the interval* $[\underline{\theta},\theta_s]$ *trade at price* $\theta_s = v\underline{\theta}/(2-v)$*, while sellers with quality higher than* θ_s *never trade.*

We refer to the stationary equilibrium outcome as the '*repeated static outcome*'.

If the repeated static outcome were the only equilibrium of our dynamic model, our analysis would not have provided much in terms of additional insight. As it turns out, however, there are many other equilibria. These equilibria are non-stationary with prices and the range of quality traded changing over time.

Example 1

Consider the following parameter values: $\underline{\theta} = 10, \overline{\theta} = 16$, $\delta = 1/9$, $v = 32/27$. In the one-period version of our model, only goods of quality lying in the interval $[10, \theta_s]$ can be traded at price θ_s, where $\theta_s = 14.545$ (approx.). As indicated above, the infinitely repeated version of this outcome is the unique stationary equilibrium of our dynamic model. There are, however, other non-stationary equilibria. The following is an equilibrium with two-period cycles. In period 1, sellers with quality lying in the interval [10,14] trade at price $p_1 = 12v = 14.22$. In period 2, all sellers entering in that period and all sellers with unsold goods from period 1 sell at price $p_2 = 13.5v = 16$. Note that the seller with quality 14 is indifferent between selling in periods 1 and 2:

$$p_1 - 14 = \delta[p_2 - 14].$$

There are no untraded goods at the end of period 2. The process can repeat itself in periods 3 and 4, and so forth. In this equilibrium, goods of all possible qualities are traded within at most two periods of entering the market. Sellers of quality below 14 sell instantaneously, while those with higher quality may wait for one period. Observe that the price and the range of quality of traded goods in the first period of each phase of the cycle is actually lower than that in the repeated static outcome. In other words, low-quality sellers selling in odd-numbered periods actually earn a lower surplus than they would in the repeated static outcome. ■

The example outlined above illustrates the possibility of a cyclical equilibrium where all goods entering the market are traded within a fixed number of periods, with the prices and average quality increasing until all potentially tradable goods are traded. In some sense, this is the simplest and most intuitive kind of non-stationary equilibrium where all goods are eventually traded. The main result of this paper is that such an equilibrium always exists.

PROPOSITION 3. *There exists a T,* $1 < T < \infty$, and *a dynamic equilibrium with T-period cycles where goods of all qualities are traded within T periods of entering the market. In particular, there exists a partition* $\{\theta_0,\theta_1,\ldots,\theta_T\}$ *of the support of the distribution of quality,* $\underline{\theta} = \theta_0 < \theta_1 < \ldots < \theta_T = \overline{\theta}$, *such that in the ith period of each cycle,* $1 \le i \le T$, *all sellers entering the market in that period who own goods of quality less than* θ_i *as well as all sellers who entered in the previous* $(i-1)$ *periods and own goods of quality lying between* θ_{i-1} *and* θ_i, *sell in period i. Prices and average quality increase along each phase of the T-period cycle.*

The basic structure of the equilibrium described in proposition 3 is as follows. As this is an equilibrium with stationary cycles, it is sufficient to describe the first phase, $t = 1,\ldots T$. Let $\eta^t(x,y)$ denote the conditional expectation of quality traded when it is known that the set of sellers is exactly equal to the set of all new entrants with quality below y and all sellers from the previous $(t-1)$ cohorts whose qualities lie between x and y. In the equilibrium described in the proposition, the average quality of goods traded in period t is $\eta^t(\theta_{t-1},\theta_t)$, where

$$\eta^t(\theta_{t-1},\theta_t) = \frac{t(\theta_t - \theta_{t-1})\frac{\theta_t+\theta_{t-1}}{2} + (\theta_{t-1} - \underline{\theta})\frac{\underline{\theta}+\theta_{t-1}}{2}}{t(\theta_t - \theta_{t-1}) + (\theta_{t-1} - \underline{\theta})}. \tag{4}$$

Indeed, $\eta^t(\theta_{t-1},\theta_t)$ can be thought of as the weighted average of quality of two sets of goods viz., the set of goods belonging to new entrants in period t and the set of goods of sellers who entered in the previous t time periods; quality is distributed uniformly in each set. As t increases, the weight of the latter set becomes more dominant.

In every period, price equals buyers' valuation of the average quality traded in that period; hence, price p_t in period t is given by $p_t = v\eta^t(\theta_{t-1},\theta_t)$. The price in each period must be at least as large as the reservation price of the highest quality sold in that period. Let s_t denote the surplus earned by the seller with the highest quality sold in period t. Then, $s_t \equiv v\eta^t(\theta_{t-1},\theta_t) - \theta_t$, and this must be non-negative in every period $t = 1,\ldots T$.

Sellers of quality below θ_t, who have not sold earlier, strictly prefer to sell in the tth period. For any $t < T$, the seller with the highest quality sold in period t must be indifferent between selling in period t and selling in period $(t+1)$. This is because sellers with qualities just above θ_t strictly prefer to sell in period $t+1$, while sellers just below θ_t strictly prefer to sell in period t. Thus,

$$s_t = \delta[v\eta^{t+1}(\theta_t,\theta_{t+1}) - \theta_t], \qquad t = 1,\ldots,T-1. \tag{5}$$

For a given θ_t, this indifference equation defines the next marginal seller θ_{t+1}. One implication of this is that in every period except for the last, the seller with marginal quality θ_t earns strictly positive surplus.

Appendix A contains the proof of the proposition. It shows the existence of a finite sequence $\{\theta_t\}_{t=0,...T}$ such that the fundamental incentive condition holds:

$$0 \leq v\eta^t(\theta_{t-1},\theta_t) - \theta_t = \delta[v\eta^{t+1}(\theta_t,\theta_{t+1}) - \theta_t], \qquad t = 1,\ldots,T-1.$$

The proof defines the points θ_t's iteratively. The basic argument is that (see lemmas 1 and 2) if marginal quality in the first period θ_1 is close enough to θ_s (so that the initial surplus earned by the marginal seller is arbitrarily small), one can iteratively define θ_ts for an arbitrarily large number of time periods (using the above incentive conditions) and ensure that these θ_ts themselves are arbitrarily close to θ_s. The next part of the proof (lemmas 3 and 4) works on the assumption that sufficient number of time periods have gone by and defines a backward iterative process where the full interval of the qualities, including $\overline{\theta}$, is traded in a certain period and then tries to iteratively construct intervals with lower marginal quality that could be traded in previous periods so as to meet the above-mentioned incentive condition. It can be shown that in this backward iteration, one can reach an interval where the highest quality lies in a neighbourhood of θ_s within a finite number of steps. Finally, the proof shows that the two paths of iteratively defined marginal qualities – one going upwards by forward iteration and the other going downwards by backward iteration – can be actually linked without violating the incentive condition.

It is possible that, apart from equilibria where all qualities are eventually traded, there are other non-stationary equilibria where the range of quality traded in the market is actually lower than $\overline{\theta}$. Consider a slight modification of example 1 where $\overline{\theta} > 16$ (all other parameter values remain unchanged). It can be checked that the equilibrium with two-period cycles described in that example is also an equilibrium here, even though the highest quality exceeds 16. In period 1, the interval of quality traded is [10,14], while in period 2, all sellers who entered in period 1 with quality lying in the interval (14,16] as well as all new entrants with quality in the interval [10,16] trade. The price in period 2 is 16 so that the seller of the highest quality traded in period 2 earns zero surplus. The process repeats itself in periods 3 and 4, and so forth. Sellers with quality lying between 16 and $\overline{\theta}$ find that the prices are always below their reservation value.

In example 1, the seller of quality $\overline{\theta}$ makes zero surplus. In general, however, it is possible that in equilibria where all goods are eventually traded, the seller of quality $\overline{\theta}$ earns zero surplus. This contrasts with the fact that in any dynamic equilibrium where the highest quality sold over all periods is some $\theta^* < \overline{\theta}$, the seller of the highest quality traded (ever) must earn zero surplus. For if the seller of quality θ* earned strictly positive surplus, there would be incentives for sellers of quality just above θ* to trade, in which case θ* would not be the highest quality ever traded.

It is also not necessary that for *every* θ^* lying strictly between θ_s and $\overline{\theta}$, there is an equilibrium where the highest quality traded (over all periods of a cycle) is exactly equal to θ^*. The restriction that the seller of quality θ^* earns exactly zero surplus in the period in which he trades is an additional restriction not required in an equilibrium path where the highest quality traded equals $\overline{\theta}$, since in that case there is no higher-quality seller who must be prevented from trading. This additional restriction reduces the possibility of equilibria with θ^* lying strictly between θ_s and $\overline{\theta}$.

All the non-stationary equilibria discussed in the examples and in proposition 3 are cyclical equilibria where, within each cycle, prices and marginal quality sold increase until they reach a certain level. However, there may also be cyclical equilibria where each cycle is non-monotonic; that is, the nature of the equilibrium path is more non-linear than the equilibrium alluded to in proposition 3.

Example 2

Consider the following parameter values: $\underline{\theta} = 10$, $\overline{\theta} = 16$, $\delta = 0.1$, $v = 1.2$. In the one-period version of our model, only goods of quality lying in the interval $[10, \theta_s]$ can be traded at a price equal to θ_s, where $\theta_s = 15$. It can be checked that the following is an equilibrium with three-period cycles, where all goods entering the market are traded within three periods. In period 1, all sellers with quality in the interval $[10, \theta_1]$ trade, where $\theta_1 = 14\frac{632}{687}$, the price in period 1 is $14\frac{218}{229}$. In period 2, *only* sellers entering in period 2 with quality in the interval $[10, 13\frac{659}{687}]$ trade; the price in period 2 is $14\frac{86}{229}$. In period 3, all sellers who entered in periods 1 and 2 but did not sell as well as all sellers entering in period 3 trade (the highest quality traded is 16). The price in period 3 is $18\frac{28}{229}$. Observe that the price as well as the marginal quality sold decline when we go from period 1 to period 2 and then increase when we go from period 2 to period 3. There are no unsold goods at the end of period 3 and the cycle can repeat itself. The prices are such that the seller with the marginal quality traded in period 1, that is, θ_1, is indifferent between trading in periods 1 and 3; that is,

$$(p_1 - \theta_1) = \delta^2(p_3 - \theta_1), \text{ where } p_1 = v\frac{\underline{\theta} + \theta_1}{2}$$

$$\text{and } p_3 = \frac{v}{2}\frac{3\overline{\theta}^2 - \theta_2^2 - \theta_1^2 - \underline{\theta}^2}{(3\overline{\theta} - \theta_2 - \theta_1 - \underline{\theta})}.$$

At the same time, the seller of the marginal quality traded in period 2 i.e., θ_2, is indifferent between trading in periods 2 and 3; that is, $(p_2 - \theta_2) = \delta(p_3 - \theta_2)$, where $p_2 = v(\underline{\theta} + \theta_2/2)$. This results in a system of two non-linear equations with two unknowns, and the above mentioned numbers are a solution to this system. Finally, observe that the seller of the highest quality sold in this equilibrium, that is, $\overline{\theta}$, earns strictly positive surplus. ■

The non-stationary equilibria we have talked about so far exhibit one consistent property, viz., that prices and the range of quality traded (or equivalently, the marginal quality traded) fluctuate over time in a cyclical fashion. Adverse selection and the entry of new goods (even when the distribution of entering quality is identical over time) combine to create non-monotonicity in market dynamics. In fact, some kind of non-monotonic behaviour is actually characteristic of *all* non-stationary equilibria.

PROPOSITION 4. *In any non-stationary equilibrium, the sequence of marginal quality sold over time is non-monotonic.*

If a monotonic equilibrium were to exist, buyers would eventually face almost no uncertainty about quality available on the market, and adverse selection in the market would asymptotically disappear. The above proposition indicates that this does not happen. Non-monotonic fluctuation in market prices and qualities of goods sold is a manifestation of the *lemons problem* in durable goods market with entry. Note that proposition 4 also implies that *the repeated static outcome (i.e., the unique stationary equilibrium) is the unique monotonic equilibrium.*

We have seen that in some of the non-stationary equilibria goods of all possible quality (including $\overline{\theta}$) are traded, and in others the highest quality traded is lower than $\overline{\theta}$. However, a consistent property of all the non-stationary equilibria described above is that the range of quality eventually traded in the market is always higher than that in the repeated static outcome; that is, the highest quality eventually traded always exceeds θ_s. Hence, the possibility of intertemporal waiting ensures that the problem of not being able to trade higher qualities, which is the *lemons problem* in the static Akerlof-Wilson framework, is less severe in the dynamic model.

PROPOSITION 5. *In any non-stationary equilibrium, the highest quality traded in the market (supremum of quality traded over all periods) is strictly greater than θ_s.*

The main argument underlying this proposition is as follows. The highest quality ever traded, say θ', can never be strictly below θ_s, since the seller of quality θ' would then earn strictly positive surplus, creating incentives for sellers of qualities higher than θ' to trade. As we are looking at non-stationary equilibria, the only other possibility that we need to rule out is that all marginal qualities lie below θ_s and converge to it in the limit. In that case, buyers would expect the quality sold to be approximately θ_s in the long run (the mass of new entrants become unimportant) and so would be willing to pay approximately $v\theta_s$. Marginal surplus would be bounded away from zero, creating incentives for sellers with quality slightly above θ_s to trade.

The length of waiting time before 'higher'-quality sellers sell in the market depends on the rate at which the equilibrium prices increase over time. The latter, in turn, is constrained by the fact that between any two successive periods in which trade takes place, prices must be such that the seller trading the marginal quality must be indifferent between selling in either of the two periods. The higher the value of δ (the lower the impatience), the smaller the rate at which equilibrium prices can increase between any two given time periods. Loosely speaking, we would expect that in any equilibrium, the length of time needed for goods of very high quality to be traded is likely to increase with δ and, in fact, becomes infinitely large as $\delta \uparrow 1$.

For any given $\delta \in [0,1)$ and $\theta \in [\theta_s,\overline{\theta}]$, let $\tau(\delta,\theta)$ denote the earliest time period *across all equilibria* in which a seller with quality θ entering in period 1 sells her good (over all equilibria).

PROPOSITION 6: *For any $\theta^* \in (\theta_s,\overline{\theta}]$, $\tau(\delta,\theta^*) \to \infty$ as $\delta \uparrow 1$, that is, as traders become infinitely patient, the length of time before a seller with any given quality higher than the static outcome θ_s can trade becomes infinitely large.*

Note that $\tau(\delta,\theta^*) \to \infty$ also implies that for a trader of quality θ^* of *any* cohort, the length of time before she can trade must become infinitely large (as $\delta \uparrow 1$).

For any *fixed t*, it is, in general, very difficult to characterize the range of quality that can be traded within t periods of a cohort's entry into the market. The only exception is when $\delta = 0$; in that case, it is easy to characterize the range of quality that can be traded within a given number of periods (for some dynamic equilibrium). That is because the necessary and sufficient condition for a dynamic equilibrium in this case is that the highest quality seller in each period should earn zero surplus. Let $z_1 = \theta_s$ and for $t > 1$, let us iteratively define z_t as the highest quality earning zero surplus in period t if all sellers with quality until z_{t-1} have traded by the end of period $(t-1)$; that is,

$$vE(z_{t-1},z_t) = z_t \text{ or } \frac{v}{2}\frac{tz_t^2 - (t-1)z_{t-1}^2 - \underline{\theta}^2}{tz_t - (t-1)z_{t-1} - \underline{\theta}} = z_t.$$

Then, for $\delta = 0$, there is a cyclical equilibrium where the marginal quality in the t-th period of the cycle is z_t (provided it does not exceed $\overline{\theta}$, at which point a new cycle begins). In this equilibrium, any seller with quality $\theta \leq z_t$ trades within t periods of entering the market.[16]

Unfortunately, no such definite characterization can be provided for $\delta > 0$, where the dynamics of quality traded over time is primarily determined by the

16 Note that z_t corresponds to the highest equilibrium quality in a one-period model when the static distribution of quality in the market is identical to that generated by a new cohort added to $(t-1)$ previous cohorts with quality larger than z_{t-1}.

intertemporal no-arbitrage condition which makes it much more difficult to pin down the highest quality traded within a fixed number of periods.

It is easy to show that, as $\delta \uparrow 1$, the highest quality traded within any *given t* periods after entry converges to θ_s. We do not state this result formally, since it is, in fact, similar in content to the statement of proposition 6.

The last feature of non-stationary equilibria we want to highlight follows from the fact that in all such equilibria, the highest quality traded in the first period must be strictly below θ_s. If the marginal quality sold is higher than θ_s, then its seller would earn negative surplus. On the other hand, if the marginal quality sold in period 1 is exactly θ_s, then the marginal seller earns zero surplus, which would imply that $\theta_t = \theta_s$ for all t which is the (stationary) repeated static equilibrium. The significance of this is that the surplus earned by sellers in the first period of any non-stationary equilibrium is lower than what they would obtain in the repeated static outcome. This, in turn, indicates why non-stationary equilibria – even those in which goods of *all* quality are sold – are not Pareto-superior to the repeated static outcome where the set of goods eventually traded is smaller and of lower quality. Further, if we consider non-stationary equilibria, where the loss in surplus in the first periods of trade (relative to the static outcome) is small, that is, θ_1 is close to θ_s, then the discounted surplus gain obtained by future trading of higher qualities is also small, because qualities significantly higher than θ_s can be traded only in periods far into the future.

4. Discussion and conclusion

Since the publication of Akerlof's seminal paper on the 'market for lemons' more than thirty years ago, economists have recognized that when sellers and buyers have correlated valuations, and sellers' valuations are unknown to buyers, market failures (in an otherwise perfectly competitive market) occur. In particular, when valuations depend on quality of goods and the market is static, market failure manifests itself in the fact that higher-quality goods cannot be traded, despite the potential gains from trade. As we have seen, this is also one of the forms in which adverse selection may manifest itself in dynamic markets; for there exist dynamic equilibria where goods of very high quality are never traded. Our analysis, however, shows that in dynamic markets there also exist equilibria where all sellers, no matter how high the quality of their good, may be able to trade in finite time. In such equilibria, the inefficiencies caused by asymmetric information manifest in the fact that sellers of higher qualities need to wait longer than sellers of lower quality in order to sell. The cost of waiting is an important factor that must be considered in any assessment of the loss in welfare caused by adverse selection.

There is a wide variety of possible equilibria, and, except for the unique stationary outcome (which is a repeated version of the equilibrium in the static model), all equilibria exhibit non-monotonic dynamics in prices and quality traded. Moreover, in all of these equilibria, a wider range of quality is traded compared with the outcome in the static model. The coordination problems associated with multiplicity of equilibria and the nature of market fluctuations generated along the equilibrium path of the non-stationary equilibria indicate that any list of problems caused by asymmetric information in dynamic durable goods markets ought to include market volatility.

We have restricted the analysis in the body of the paper to the case of perfectly durable goods. However, we can allow for depreciation. For example, consider the following specific form of depreciation: in every period there is a chance $\lambda \in [0,1]$ that the good does not survive until the next period. In that case the incentive equation (5) can be written as

$$s_t = \delta[(1-\lambda)(v\eta^{t+1}(\theta_t,\theta_{t+1}) - \theta_t)) + \lambda \cdot 0] = \delta(1-\lambda)[(v\eta^{t+1}(\theta_t,\theta_{t+1}) - \theta_t)].$$

This type of depreciation is equivalent to a change in the discount factor and does not change the qualitative properties of our equilibrium analysis.

Our analysis in this paper has been confined to the role of the price mechanism as an institution for organizing resource allocation under asymmetric information. While the qualitative predictions of our model are important for an understanding of the market as a trading institution, observed market dynamics may differ considerably from our predictions. This is because in the real world, the price mechanism is augmented by other non-market institutions and technologies that enable agents to signal or screen information and they alter the behaviour of agents as well as the pattern of trade.

Appendix A: Proof of Proposition 3

For any given θ_1 we define the process $\{\theta_t\} = \{\theta_t(\theta_1)\}$ with $\theta_{t+1} > \theta_t$, if it exists, iteratively by the indifference equation (5):

$$\frac{v}{2}\frac{(t+1)\theta_{t+1}^2 - t\theta_t^2 - \underline{\theta}^2}{(t+1)\theta_{t+1} - t\theta_t - \underline{\theta}} = \theta_t + \frac{s_t}{\delta}, \tag{A0}$$

where the L.H.S. is the expression for p_{t+1}. Moreover, if the process is well defined up to period t, then we can define the surplus of the marginal seller in a period by $s_t = p_t - \theta_t$. The above implies that θ_t and s_t are functions of θ_1. Sometimes we take the liberty of dropping this dependence and write simply θ_t and s_t instead of $\theta_t(\theta_1)$ and $s_t(\theta_1)$. Sometimes we also write $s_t(\theta_t)$ instead of $s_t(\theta_1)$.

From the statement of proposition 3 it is clear that in order to have a T-period equilibrium cycle, we have to construct a sequence of marginal qualities $\{\theta_t\}_{t=1}^{T}$ such that indifference equation (5) is satisfied, $\theta_T = \overline{\theta}$, and the seller of marginal quality makes non-negative surplus in every period. In principle, one could try to construct such a sequence in several ways. One possibility is to start with an arbitrary θ_1 lying below the highest quality sold in the static model and then iteratively define a 'forward process' $\theta_t > \theta_{t-1}$, $t > 1$ using the indifference equation (5), verify that the buyers' valuation of the average quality on each interval traded exceeds the highest quality and hope that in a finite number of steps, the entire support of the distribution of quality will be traded, thus completing a phase of the cyclical equilibrium. Another possibility would be to fix some T, define an arbitrary θ_{T-1}, and then iteratively define a 'backward process' $\theta_{t-1} < \theta_t$, $t-1 < T-1$ and hope that one can cover the entire support in T-steps.

Both approaches run into problems. It is difficult to ensure in the forward-looking process that surpluses of marginal qualities remain non-negative when the marginal quality is sufficiently higher than θ_s. In the backward-looking process the problem is to ensure that a marginal quality satisfying the indifference equation exists; the surplus earned by the marginal seller θ_t after some $T-t$ steps of the iteration (for t large) may be so large that there is no $\theta_{t-1} < \theta_t$ that 'makes θ_t indifferent.'

The structure of our proof is therefore as follows. We define two sequences, one by the forward-looking process and one by the backward-looking process and then combine these sequences into one large sequence that satisfies all the equilibrium properties. The first two lemmas define the forward-looking sequence starting from a small left-neighbourhood of θ_s. According to lemma 1, if θ_1 is chosen arbitrarily close to θ_s, then the upward sequence of marginal qualities θ_t can be defined (while satisfying the indifference of marginal seller and the non-negative surplus conditions) for a finite but arbitrarily large number of periods, and the surpluses that the marginal qualities make grow at a rate that is higher than and bounded away from 1. Lemma 2 establishes some more properties of this finite sequence. As θ_t is the largest quality sold in period t, it is clear that the average quality in that period cannot be larger than θ_t, and, therefore, the surplus s_t of this marginal seller cannot be larger than $(v-1)\theta_t$. Lemma 2 says that if the process is defined for a sufficiently large t (by choosing θ_1 arbitrarily close to θ_s), then of all the θ_1s for which the forward sequence can be defined for (any such large) t periods, there is at least one θ_1 such that the surplus s_t of the marginal seller θ_t is arbitrarily close to $(v-1)\theta_t$, and, moreover, such θ_t is itself arbitrarily close to θ_s.

LEMMA 1. *Choose any* α *with* $1 > \alpha > \delta v/[\delta v + 2(1-\delta)(v-1)]$. *For any* $\tau > 1$, *define* $\underline{\theta}_1(\tau) \equiv inf\{x | \theta_1 \in (x, \theta_s)$ *and* $t \leq \tau \Rightarrow s_t > 0$ *and* $\alpha s_t > s_{t-1}\}$. *For any* τ, $\underline{\theta}_1(\tau) < \theta_s$ *and* $\lim_{\tau\to\infty} \underline{\theta}_1(\tau) = \theta_s$.

Proof. We first show that for any given value of θ_t, equation (A0) uniquely defines θ_{t+1}. We then solve (A0) to get an expression for this solution of θ_{t+1}. Using this solution, we then use an induction argument where we neglect terms involving $\epsilon_t^2, \epsilon_t s_t, s_t^2$, since we know that these terms are very small when we choose θ_1 close enough to θ_s. Write $\theta_t = \theta_s + \epsilon_t$. For a given θ_t equation (A0) implicitly defines θ_{t+1} in the following way. For a given θ_t, p_{t+1} is defined by and increasing in θ_{t+1}, and at $\theta_{t+1}=\theta_t$ L.H.S. equals $v/2(\theta_t+\underline{\theta})$. As $s_1 = v/2(\theta_1+\underline{\theta})-\theta_1$, it is easy to see that at $\theta_2=\theta_1$, L.H.S. is strictly smaller than R.H.S. Moreover, for θ_1 close to θ_s, s_1 is close to 0. So for a given θ_1 close to θ_s, (A0) defines $\theta_2>\theta_1$. More generally, at $\theta_{t+1}=\theta_t$ L.H.S. of (A.0) is strictly smaller than R.H.S. when $\underline{\theta}<\frac{2-v}{v}\theta_t+\frac{2}{\delta v}s_t$, which is the case as $\theta_t>\theta_1$ and $s_t>s_1$.

Solving (A0) for θ_{t+1} gives

$$\theta_{t+1}=\frac{(t+1)\left[\frac{v}{2-v}\underline{\theta}+\epsilon_t+\frac{s_t}{\delta}\right]+\sqrt{D}}{v(t+1)}, \tag{A1}$$

where

$$D=\underline{\theta}^2\frac{v^2(t+1)^2(v-1)^2}{(2-v)^2}+2\epsilon_t\underline{\theta}(t+1)v\frac{(v-1)(vt-t+1)}{2-v} -\frac{s_t\underline{\theta}}{\delta}2v(t+1)\frac{(v-1)(t-1)}{2-v}-2v(t+1)[(1-\frac{v}{2})\epsilon_t^2+\frac{\epsilon_t s_t}{\delta}].$$

The induction argument proceeds in three steps. First, we will show that

$$\left[vt\left(1-\frac{\delta}{\alpha}\right)+v-2-\frac{\delta}{\alpha}v\right]s_i>\delta(2-v)\epsilon_i \text{ holds for } i=1.$$

Next, we show for a fixed τ that there exists ϵ_1 small enough such that for any $t<\tau$ if the above inequality holds for $i=t$, then it also holds for $i=t+1$. Finally, we show that if the above inequality holds, then it is also true that $\alpha\, s_{t+1}>s_t$. This implies that $s_t>0$ for all $t<\tau$.

(1) We first prove that $[vt(1-(\delta/\alpha))+v-2-(\delta/\alpha)v]s_t>\delta(2-v)\epsilon_t$ holds for $t=1$ and that $s_1>0$. Given that at $t=1$, quality is normally distributed $s_1=-(1-v/2)\epsilon_1$. As $\epsilon_1<0$, it is easy to see that $s_1>0$. For $t=1$, the inequality above reduces to $-2(v-1-(\delta/\alpha)v)((2-v)/2)\epsilon_1>\delta(2-v)\epsilon_1$. As $\epsilon_1<0$, this can be rewritten as $-2(v-1-\delta/\alpha v)<2\delta$. This inequality holds for all $\alpha>\delta v/[\delta v+2(1-\delta)(v-1)]$.

(2) We next argue that $[vt(1-(\delta/\alpha))+v-2-(\delta/\alpha)v]s_t>\delta(2-v)\epsilon_t$ implies

$$\left[v(t+1)\left(1-\frac{\delta}{\alpha}\right)+v-2-\frac{\delta}{\alpha}v\right]s_{t+1}>\delta(2-v)\epsilon_{t+1}=\delta(2-v)(\theta_{t+1}-\theta_s).$$

When ϵ_1 is small enough, ϵ_t and s_t are also small enough so that we can neglect terms involving ϵ_t^2 and $\epsilon_t s_t$. Given (A1), a sufficient condition for this inequality to hold is then

$$\frac{v(t+1)(1-\frac{\delta}{\alpha})+\frac{1}{v}(v-2)(v+1)-\frac{\delta}{\alpha}v}{\delta(2-v)}s_t-\frac{1}{v}\epsilon_t >\left(1-\frac{1}{v}\right)\epsilon_t+\left(\frac{1}{\delta}-\frac{1}{\delta v}-1\right)s_t,$$

which can be rewritten as

$$\frac{v(t+1)(1-\delta/\alpha)-\delta v/\alpha+(2-v)(\delta-2)}{\delta(2-v)}s_t > \epsilon_t.$$

Given the induction hypothesis, this inequality holds if

$$v(t+1)\left(1-\frac{\delta}{\alpha}\right)-\frac{\delta}{\alpha}v+(2-v)(\delta-2) > vt\left(1-\frac{\delta}{\alpha}\right)+v-2-\frac{\delta}{\alpha}v.$$

This inequality holds if $\alpha > \delta v/[\delta v + 2(1-\delta)(v-1)]$, which is assumed in the lemma.

(3) Finally, we prove that $[vt(1-(\delta/\alpha))(+v-2-(\delta/\alpha))v]s_t > \delta(2-v)\epsilon_t$ implies that $\alpha s_{t+1} > s_t$. The latter inequality is equivalent to $\alpha(p_{t+1}-\theta_{t+1}) > s_t$, or

$$\theta_{t+1} < \theta_t + \left(\frac{1}{\delta}-\frac{1}{\alpha}\right)s_t. \tag{A2}$$

Given (A1), this can be rewritten as

$$\frac{v-1}{2-v}\underline{\theta}+\left(1-\frac{1}{v}\right)\epsilon_t+\left(\frac{1}{\delta}-\frac{1}{\delta v}-\frac{1}{\alpha}\right)s_t > \frac{1}{v(t+1)}\sqrt{D}$$

or

$$\left[vt\left(1-\frac{\delta}{\alpha}\right)+v-2-\frac{\delta}{\alpha}v\right]s_t > \delta(2-v)\epsilon_t - \frac{\delta(2-v)t}{(v-1)\underline{\theta}}[(1-\frac{v}{2})\epsilon_t^2+\frac{\epsilon_t s_t}{\delta}]. \tag{A3}$$

We know that if we take ϵ_1 small enough, $s_1 = -(1-v/2)\epsilon_1$ will be small. Similarly, as ϵ_t and s_t are continuous functions of ϵ_{t-1} and s_{t-1} and as $\epsilon_t = s_t = 0$ if $\epsilon_1 = 0$, ϵ_t and s_t can be made arbitrarily small as well. In particular, for a given τ we can choose ϵ_1 so small such that the last term on the right-hand side of inequality (A3) is negligible for all $t < \tau$. Hence, if $[vt(1-(\delta/\alpha)) + v - 2 - (\delta/\alpha)v]s_t > \delta(2-v)\epsilon_t$, then $\alpha\ s_{t+1} > s_t$ and, thus, $s_t > 0$ for all $t < \tau$. Above, we have shown that for any τ there exists a left-neighbourhood of θ_s such that for all $t < \tau$ surpluses are positive and increasing by at least a factor α. Finally, we show that $\lim_{t\to\infty}\underline{\theta}_1(t) = \theta_s$. Suppose that this were not the case. Then there would be a monotonic sequence $\{\theta_t\} = \{\theta_t(\theta_1)\}$ with $\theta_{t+1} > \theta_t$ defined by (A0). On the compact interval $[\underline{\theta},\overline{\theta}]$, this sequence $\{\theta_t\}$ must then be convergent, which would then constitute a dynamic monotonic equilibrium contradicting proposition 4. So, for all θ_1 there exists a t such that $\alpha s_{t+1}(\theta_1) < s_t(\theta_1)$. ■

LEMMA 2. *For any* $\tilde{\epsilon} > 0$ *and* $\epsilon > 0$, *there exists a* t *(large enough) and a* $\hat{\theta}_1(t) < \theta_s$ *such that* $\theta_1 < \theta_t(\theta_1) < \theta_s + \tilde{\epsilon}$ *for all* $\hat{\theta}_1(t) < \theta_1 < \theta_s$ *and* $s_t(\hat{\theta}_1(t)) \geq (v-1)\theta_t(\hat{\theta}_1(t)) - \epsilon$.

Proof. From lemma 1 we know that starting from a given θ_1 close to θ_s, we can define a forward sequence for τ periods such that surpluses of marginal sellers are increasing by a factor at least equal to $1/\alpha$. In the proof of lemma 2 we first enquire for a *given* θ_1 what is the first period when surpluses stop increasing at the rate of $1/\alpha$. After lengthy calculations we conclude that this happens when the ratio of marginal qualities $\theta_{t+1}/\theta_t > z$ for some given $z > 1$. We next show in seven short steps that if this were to occur in some period t (for this given θ_1), then it must be the case that there is *some* $\hat{\theta}_1(t)$, such that the surplus made by the tth marginal seller $\theta_t(\hat{\theta}_1(t))$, in the forward-looking process starting from $\hat{\theta}_1(t)$ as the initial marginal quality, is arbitrarily close to $(v-1)\theta_t$ (the larger t is, that is, larger the initial 'given θ_1' is, the closer it is). We finally show that this marginal seller $\theta_t(\hat{\theta}_1(t))$ and all marginal sellers $\theta_t\,(\theta_1)$ for $\theta_1 > \hat{\theta}_1(t)$ are arbitrarily close to θ_s.

Choose

$$\alpha \text{ s.t.} \max\left[\frac{\delta(v^2+v-1)}{v^2+\delta v-1}, \frac{\delta v}{\delta v+2(1-\delta)(v-1)}\right] < \alpha < 1.$$

We determine when, for a given θ_1, $\alpha s_{t+1} < s_t$. We know that, when θ_1 is close to θ_s the first time that $\alpha s_{t+1} < s_t$ occurs, t must be arbitrarily large. From the proof of lemma 1 we know that this is equivalent to determining when $\theta_{t+1} > \theta_t + ((1/\delta) - (1/\alpha))s_t$. To this end, we take the indifference equation

$$\frac{v}{2}\frac{t\theta_t^2-(t-1)\theta_{t-1}^2-\underline{\theta}^2}{t\theta_t-(t-1)\theta_{t-1}-\underline{\theta}} = \theta_{t-1}+\frac{s_{t-1}}{\delta}$$

and rewrite it as

$$\theta_t = \frac{\left(\theta_{t-1}+\frac{s_{t-1}}{\delta}\right)t+\sqrt{D}}{vt}, \tag{A4}$$

where

$$D = \left[(v-1)t\theta_{t-1}-\frac{ts_{t-1}}{\delta}\right]^2 + vt\left[(2-v)\theta_{t-1}^2-2\underline{\theta}\theta_{t-1}+v\underline{\theta}^2\right]+2vt\frac{s_{t-1}}{\delta}(\theta_{t-1}-\underline{\theta}).$$

Given (A4), $\theta_{t+1} > \theta_t + ((1/\delta)-(1/\alpha))s_t$, if and only if,

$$2(v-1)(t+1)\left(\frac{1}{\delta}-\frac{1}{\alpha}\right)\theta_t s_t + (t+1)\left(\frac{1}{\delta}-\frac{1}{\alpha}\right)s_t^2\left[\frac{v-2}{\delta}-\frac{v}{\alpha}\right] < (\theta_t-\underline{\theta})\left[(2-v)(\theta_t-\theta_s)+2\frac{s_t^2}{\delta}\right]. \tag{A5}$$

As (A5) is quadratic in s_t, there are two solutions for which (A5) holds with equality. We will denote these solutions by $\underline{s}(\theta_t)$ and $\bar{s}(\theta_t)$, respectively, and $\underline{s}(\theta_t) < \bar{s}(\theta_t)$. It is easy to see that for a given t, (A5) holds for $s_t < \underline{s}(\theta_t)$

and $s_t > \bar{s}(\theta_t)$. Note that for t large enough, $\underline{s}(\theta_t)$ is close to zero and positive. We will show that when t is large enough, it cannot be the case that $s_t < \underline{s}(\theta_t)$, so that effectively $s_t > \bar{s}(\theta_t)$ has to hold when $\alpha s_{t+1} < s_t$. To do so, suppose θ_1 is such that for t periods $\alpha s_k > s_{k-1}$, $k = 2,..,t$ for some $\alpha < 1$. The indifference equation $s_{t-1} = \delta(p_t - \theta_{t-1})$ can be rewritten as $\theta_t - \theta_{t-1} = -s_t + (1/\delta)s_{t-1}$.

Iteratively applying this indifference equation yields

$$\theta_t - \theta_1 = \frac{1}{\delta}(s_1 - s_t) + \left(\frac{1}{\delta} - 1\right)\sum_{k=2}^{t} s_k. \tag{A6}$$

As $s_1 < s_t$ and $\theta_1 < \theta_s$, this implies that $\theta_t - \theta_s < ((1/\delta - 1))\Sigma_{k=2}^{t} s_k$. As, moreover, $\alpha s_k > s_{k-1}$, this in turn implies that

$$\theta_t - \theta_s < \left(\frac{1}{\delta} - 1\right) s_t \sum_{k=0}^{t-2} \alpha^k = \left(\frac{1}{\delta} - 1\right) s_t \frac{1 - \alpha^{t-1}}{1 - \alpha} < \left(\frac{1}{\delta} - 1\right) \frac{s_t}{1 - \alpha}. \tag{A7}$$

For large t, and using (A7), (A5) holds true only if

$$2(v-1)\theta_t + s_t\left(\frac{v-2}{\delta} - \frac{v}{\alpha}\right) < \frac{\theta_t\left[\frac{1-\delta}{\delta(1-\alpha)}\right] + 2\frac{s_t}{\delta}}{(t+1)\left(\frac{1}{\delta} - \frac{1}{\alpha}\right)}. \tag{A8}$$

As θ_t and s_t are bounded, the R.H.S. is arbitrarily small for large t. Since, for large t, $\underline{s}(\theta_t)$ is close to zero, it is easy to see that (A5) cannot hold for $s_t < \underline{s}(\theta_t)$. Hence, (A5) can hold only for $s_t(\theta_1) > \bar{s}(\theta_t)$. For large t (A8) can be rewritten as

$$\theta_t + \frac{s_t}{\delta} > \frac{\left(1 + \frac{\delta}{\alpha}\right)v\theta_t}{2 - \left(1 - \frac{\delta}{\alpha}\right)v} - o(\epsilon).$$

Substituting this back into (A4) gives a condition on θ_{t+1} in terms of θ_t:

$$\theta_{t+1} > \frac{1 + \frac{\delta}{\alpha}}{2 - \left(1 - \frac{\delta}{\alpha}\right)v}\theta_t + \frac{1}{v(t+1)}\sqrt{D^1} - o(\epsilon), \tag{A9}$$

where

$$D^1 = (t+1)^2\theta_t^2 \frac{[v(v-1)(1-\delta) + \delta(1 - \frac{1}{\alpha})v^2]^2}{[2 - (1-\delta)v]^2} + v(t+1)(\theta_t - \underline{\theta})[\theta_t(2-v) - v\underline{\theta}]$$
$$+ v(t+1)\frac{4(v-1) + 2\delta v(1 - \frac{1}{\alpha})}{2 - (1-\delta)v}\theta_t(\theta_t - \underline{\theta}).$$

For large t, (A9) reduces to

$$\theta_{t+1} > \frac{\left(1+\frac{\delta}{\alpha}\right)\theta_t}{2-\left(1-\frac{\delta}{\alpha}\right)v} + \frac{[v(v-1)(1-\delta)+\delta(1-\frac{1}{\alpha})v^2]\theta_t}{2-\left(1-\frac{\delta}{\alpha}\right)v} - o(\epsilon).$$

Define

$$z = \left[\left(1+\frac{\delta}{\alpha}\right) + v(v-1)(1-\delta) + \delta\left(1-\frac{1}{\alpha}\right)rv^2\right] \Big/ \left[2-\left(1-\frac{\delta}{\alpha}\right)v\right].$$

Straightforward calculations show that

$$z > 1 \text{ if } \frac{\delta}{\alpha} < \frac{v^2+\delta v-1}{v^2+v-1},$$

which, given the restriction imposed on α holds. Hence, $\alpha s_{t+1} < s_t$ only if $\theta_{t+1} > z\theta_t$ for some $z > 1$.

We use this result to show that for any $\epsilon > 0$ there exist a t and $\hat{\theta}_1(t)$ such that $s_t(\hat{\theta}_1(t)) \geq (v-1)\theta_t(\hat{\theta}_1(t)) - \epsilon$. This part of the proof proceeds in seven steps:

(i) We first define $\theta^*_{t+1}(\theta_t)$ as the θ_{t+1} that receives maximum surplus in $t+1$ given θ_t. For a given θ_t, surplus in $t+1$ is given by

$$s_{t+1} = \frac{v}{2}\frac{(t+1)\theta^2_{t+1} - t\theta^2_t - \underline{\theta}^2}{(t+1)\theta_{t+1} - t\theta_t - \underline{\theta}} - \theta_{t+1}.$$

Taking the first derivative, and setting it equal to 0, gives

$$\frac{\partial s_{t+1}}{\partial\theta_{t+1}} = \frac{v}{2}\frac{2(t+1)\theta_{t+1}[(t+1)\theta_{t+1} - t\theta_t - \underline{\theta}] - (t+1)\left[(t+1)\theta^2_{t+1} - t\theta^2_t - \underline{\theta}^2\right]}{[(t+1)\theta_{t+1} - t\theta_t - \underline{\theta}]^2} - 1 = 0.$$

This can be rewritten as

$$\theta^*_{t+1} = \frac{t\theta_t + \underline{\theta} + (\theta_t - \underline{\theta})\sqrt{\frac{vt}{2-v}}}{t+1}. \tag{A10}$$

Note that for large t, $\theta^*_{t+1} \to \theta_t$. The maximal surplus in period $t+1$ is then given by

$$\begin{aligned} s^*_{t+1} &= \frac{t\theta_t + \underline{\theta} - (\theta_t - \underline{\theta})\sqrt{vt(2-v)}}{t+1} \\ &= (v-1)\theta_{t+1} - (\theta_t - \underline{\theta})\frac{(v-1)\sqrt{\frac{vt}{2-v}} + \sqrt{vt(2-v)}}{t+1} \end{aligned}$$

Note that as $t \to \infty$, $s^*_{t+1} \to (v-1)\theta_{t+1}$. So, for a fixed small $\epsilon > 0$ and any t large enough, s_{t+1} can in principle get larger than $(v-1)\theta_{t+1} - \epsilon$.

(*ii*) (A10) implies that for any t large enough, $(\theta^*_{t+1}/\theta_t) < z$.
(*iii*) From the above we know that for given large t there exists a $\theta_1 < \theta_s$ with $\theta_{t+1}(\theta_1) > z\theta_t(\theta_1)$ for some $z > 1$.
(*iv*) For a given large t and for all $\theta_1 \in (\underline{\theta}_1(t),\theta_s), 1 < \theta^*_{t+1}/\theta_t < z$.
(*v*) By choosing $\theta_1 = \theta_s$, we know that $\theta_{t+1}/\theta_t = 1$.
(*vi*) As θ_{t+1} and θ_t continuous in θ_1 and $\theta_t > 0$, θ_{t+1}/θ_t is also a continuous function of θ_1.
(*vii*) Applying the intermediate value theorem, we can argue that for any t large enough, there exists a $\hat{\theta}_1(t) \in (\underline{\theta}_1(t),\theta)$ such that $\theta_{t+1}(\hat{\theta}_1(t))/\theta_t(\hat{\theta}_1(t)) = \theta^*_{t+1}/\theta_t$. In case for a given t there are multiple solutions, $\hat{\theta}_1(t)$ will denote the largest of them.

So far, we have shown that for any small $\epsilon > 0$ there exists a (large) t and a $\hat{\theta}_1(t)$ such that $s_t(\hat{\theta}_1(t)) \geq (v-1)\theta_t(\hat{\theta}_1(t)) - \epsilon$. Now choose T large enough. We finally show that for any $\tilde{\epsilon} > 0$ there exists a $t > T$ such that $\theta_1 < \theta_t(\theta_1) < \theta_s + \tilde{\epsilon}$ for all $\hat{\theta}_1(t) < \theta_1 < \theta_s$. From (*i*)-(*vii*) it follows that for all $\theta_1 \in (\hat{\theta}_1(t),\theta_s), \theta_t(\theta_1) < \theta^*_t(\theta_{t-1}(\theta_1))$. So, $(\theta_t(\theta_1) - \theta_{t-1}(\theta_1)) < (\theta^*_t(\theta_1) - \theta_{t-1}(\theta_1))$ for all $\theta_1 \geq \hat{\theta}_1(t)$. Equation (A10) shows that the R.H.S. of this last inequality goes to 0 as t becomes large. Hence, for all (very small) $\epsilon_1 > 0$ and for all $\theta_1 \geq (\hat{\theta}_1(t))$ there exists a (large) t such that $\theta_t(\theta_1) - \theta_{t-1}(\theta_1) < \epsilon_1$.

We know that $s_t(\theta_1)$ is a continuous function on $(\hat{\theta}_1(t),\theta_s)$, ranging from $(v-1)\theta_t - \epsilon$ to 0. Choose any small $\epsilon_2 \in (0, (v-1)\theta_t - \epsilon)$. Observe that $s_{t-1}(\theta_1) = \delta[s_t(\theta_1) + \theta_t(\theta_1) - \theta_{t-1}(\theta_1)]$. Hence, for all $s_t \geq \epsilon_2$, we can write

$$\frac{s_{t-1}(\theta_1)}{s_t(\theta_1)} = \delta + \delta\frac{\theta_t(\theta_1) - \theta_{t-1}(\theta_1)}{s_t(\theta_1)} < \delta + \delta\frac{\epsilon_1}{\epsilon_2}.$$

By choosing ϵ_1 much smaller than ϵ_2, the R.H.S. of this expression can be made arbitrarily close to δ. To prove the claim, rewrite equation (A6) as

$$\theta_t(\theta_1) = \theta_1 + \frac{s_1}{\delta} + \left(\frac{1}{\delta} - 1\right)\sum_{j=2}^{t-k-1} s_j + \left\{\left(\frac{1}{\delta} - 1\right)\sum_{j=t-k}^{t} s_j - \frac{s_t}{\delta}\right\}.$$

For a given value ϵ_2 and for each value of θ_1 there exists a finite number $k \geq 0$, depending on θ_1, such that $s_{t-k} < \epsilon_2$. If $s_t < \epsilon_2$, k is equal to 0. On the other hand, if $s_t \geq \epsilon_2$, as s_t is finite (smaller than $(v-1)\theta_t(\theta_1(t)) < (v-1)\overline{\theta}$) and has been growing at a rate strictly larger than 1, there must be a finite k such that $s_{t-k} < \epsilon_2$. It follows that for large t, and for all $\theta_1 > \hat{\theta}_1(t)$:[17]

17 Note that here we are using the fact that surplus is always increasing at a rate larger than $1/\alpha$, $\alpha < 1$, and for $s_{t-k}(\theta_1) > \epsilon_2$ it is increasing at a rate larger than $1/[\delta(1 + (\epsilon_1/\epsilon_2))]$.

$$\theta_t(\theta_1) < \theta_1 + \frac{s_1}{\delta} + \left(\frac{1}{\delta} - 1\right)\frac{\epsilon_2}{1-\alpha} + \left\{\left(\frac{1-\delta}{\delta}\right)\frac{s_t}{1-\delta(1+\frac{\epsilon_1}{\epsilon_2})} - \frac{s_t}{\delta}\right\}. \qquad \text{(A11)}$$

As $\underline{\theta}_1(t) < \hat{\theta}_1(t)$ and $\lim_{t\to\infty}\underline{\theta}_1(t) = \theta_s$, it follows that $\lim_{t\to\infty}\hat{\theta}_1(t) = \theta_s$, and therefore there exists an ϵ_3 close to 0 such that $\theta_1 + (s_1/\delta) < \theta_s + \epsilon_3$ for all $\theta_1 > \hat{\theta}_1(t)$ and t large. Hence, there are arbitrarily small values of ϵ_1,ϵ_2,ϵ_3 such that

$$\theta_t(\theta_1) < \theta_s + \epsilon_3 + \left(\frac{1}{\delta} - 1\right)\frac{\epsilon_2}{1-\alpha} + \frac{\epsilon_1/\epsilon_2}{1-\delta(1+\frac{\epsilon_1}{\epsilon_2})}\cdot s_t$$

for all large t and all $\theta_1 > \hat{\theta}_1(t)$. As s_t is bounded, it follows that for any given t we can choose ϵ_1 much smaller than ϵ_2 in such a way that for all $\theta_1 > \hat{\theta}_1(t)$, $\theta_t(\theta_1)$ is arbitrarily close to θ_s. ■

In lemmas 1 and 2, we have some properties of the *increasing* sequence of marginal qualities defined by the 'forward-looking process.' One of the important properties we have shown is that if we choose θ_1 close enough to θ_s, then the sequence is defined for an arbitrarily large number of periods (while satisfying the conditions requiring the marginal seller to be indifferent and earn non-negative surplus). Lemmas 3 and 4 analyse the possibility of defining a *decreasing* sequence of marginal qualities through a backward-looking process for a fixed large T, where the initial point is the marginal (lowest) quality sold in the $(T-1)$th period, the next point is the marginal quality sold in the $(T-2)$th period, and so on.

More specifically, in lemma 3 we analyse a vector of decreasing marginal qualities through a backward-looking process, assuming there is *no entry of goods after period 1*. This is the 'stock' model as analysed in Janssen and Roy (2002) and is purely a benchmark case. This is very helpful, since for any given K, one can define K marginal qualities, through a backward-looking process in our 'true' model, that are arbitrarily close to the first K marginal qualities of the 'pseudo' sequence (by choosing T to be arbitrarily large).[18]

The pseudo-sequence of marginal qualities that we construct has the following properties (remember, no entry after initial period): if we choose a starting point y (in the full sequence of equilibrium values this would correspond to θ_{T-1}) in a certain interval, which we denote by $(\beta(\overline{\theta}),\overline{\theta}]$, where $\beta(x)$ is a function defined below, then we can show that there exists finite $K > 1$ such that decreasing marginal qualities $\gamma_i(y)$, $i = 1 \dots K$, are well-defined (statement iv of the lemma says that the marginal seller is indifferent and it ensures that the marginal sellers earn positive surplus), that there is a minimum distance of Δ between two marginal sellers (statement v), and that the marginal qualities are *continuous and increasing functions* of the initial starting point y.

18 Note that these K marginal qualities in our true model actually correspond to the *last* K-periods of a T-period cyclical equilibrium.

Let x_0 and x_1 be defined by

$$(v-1)\underline{\theta} = \delta[v(\frac{\underline{\theta}+x_0}{2} - \underline{\theta})]$$

and

$$v\left[\frac{\underline{\theta}+x_1}{2}\right] - x_1 = \delta(v-1)x_1.$$

It is easy to check that $\underline{\theta} < x_1 < \theta_s$ and $x_1 < x_0$. Define the function $\alpha(x)$ by

$$(v-1)\alpha(x) = \delta\left[v\left(\frac{\alpha x + x}{2}\right) - \alpha(x)\right] \quad \text{for } x \in [x_0, \overline{\theta}]$$
$$\alpha(x) = \underline{\theta} \qquad \text{for } x \in [\underline{\theta}, x_0].$$

Define the function $\beta(x)$ by

$$v\left[\frac{\beta(x)+x}{2} - x\right] = \delta(v-1)x \quad \text{for } x \in [x_1, \overline{\theta}]$$
$$\beta(x) = \underline{\theta} \qquad \text{for } x \in [\underline{\theta}, x_1].$$

Straightforward algebra shows that $\alpha(x) < \beta(x)$ for all $x > x_1$. Define the function $\omega(x) = (\alpha(x) + \beta(x))/2 > \alpha(x)$.

For $0 < x \leq y$, let $\eta(x,y) = \eta^1(x,y) = (x+y)/2$. If no entry occurs in period 1, then $\eta(x,y)$ is the average quality traded in the market in *any period* if seller types in the interval (x,y) trade. Note that $\eta(x,x) = x$ and that $\eta(x,y)$ is a smooth function.

LEMMA 3. *There exists K,* $1 \leq K \leq \infty$, *functions* $\gamma_i(y)$, $i = 1,2,\ldots,K$ *and intervals* $[a_k, b_k]$, $[a_{k-1}, b_{k-1}]$, …, $[a_1, b_1]$ *such that*

I. $a_{i+1} < a_i < b_{i+1} < b_i$, $i = 1,\ldots K-1$

II. $a_1 > \omega(\overline{\theta}), b_1 = \overline{\theta}, a_K < \theta_s, a_{K-1} > \theta_s$

III. $\gamma_i: [a_1, b_1] \rightarrow [a_i, b_i]$, $i = 1,\ldots K$

IV. γ_i *is continuous and increasing on* $[a_1, b_1]$. $\gamma_1(y) = y$, $\gamma_0(y) = \overline{\theta}$ *and for all* $y \in [a_1, b_1]$: $v\eta(\gamma_{i+1}(y), \gamma_i(y)) - \gamma_i(y) = \delta[v\eta(\gamma_i(y), \gamma_{i-1}(y)) - \gamma_i(y)]$, $i = 1,2,\ldots, K-1$

V. $\forall y \in [a_1, b_1]$ *and* $i = 2,\ldots K$: $\inf((\gamma_{i-1}(y)) - \gamma_i(y)) > [(2-v)(v-1)(1-\delta)^2/v^2]\underline{\theta} \equiv \Delta > 0$.

Proof. For any $y > \underline{\theta}, x \in [\omega(y), y]$ implies that

$$(v-1)x > \delta[v\eta(x,y) - x]. \tag{A12}$$

So that *either* there exists $z \in [\underline{\theta}, x]$ such that $v\eta(z,x) - x = \delta[v\eta(x,y) - x]$ or $v\eta(\underline{\theta}, x) - x > \delta[v\eta(x,y) - x]$. Define the functions γ_0, γ_1 on $[\omega(\overline{\theta}), \overline{\theta}]$ by $\gamma_0(y) = \overline{\theta}$ and $\gamma_1(y) = y$. From (A12) it follows that for any $y \in [\omega(\overline{\theta}), \overline{\theta}]$,

either A.I $\quad v\eta(\underline{\theta},y) - y > \delta[v\eta(y,\overline{\theta}) - y]$
or $\quad$ A.II $\quad \exists\gamma_2(y) < \gamma_1(y)$ such that

$$v\eta(\gamma_2(y),\gamma_1(y)) - \gamma_1(y) = \delta[v\eta(\gamma_1(y),\gamma_0(y)) - \gamma_1(y)]. \tag{A13}$$

If A.I holds for some $y \in [\omega(\overline{\theta}),\overline{\theta}]$, then $\exists\ \hat{y} \in [\omega(\overline{\theta}),\overline{\theta}]$, where

$$v\eta(\underline{\theta},\hat{y}) - \hat{y} = \delta[v\eta(\hat{y},\overline{\theta}) - \hat{y}]$$

Observe that $\hat{y} < \theta_s$ (otherwise $v\eta(\underline{\theta},\hat{y}) - \hat{y} \leq 0$). In case A.I set $a_1 = \hat{y}, b_1 = \overline{\theta}$, $K = 1$. If A.II holds, then (A13) holds and we can set $a_1 = \omega(\overline{\theta}), b_1 = \overline{\theta}$. One can rewrite (A13) as $\gamma_2(y) = (1 - \delta)(2 - v/v)y + \delta\overline{\theta}$, which shows that γ_2 is continuous and increasing on $[\omega(\overline{\theta}),\overline{\theta}]$. Therefore, γ_2 maps the interval $[\omega(\overline{\theta}),\overline{\theta}]$ to an interval $[a_2, b_2]$, where $a_2 = \gamma_2(\omega(\overline{\theta})), b_2 = \gamma_2(\overline{\theta}) = \beta(\overline{\theta}) > \omega(\overline{\theta})$.

Suppose we have defined $\gamma_0, \gamma_1, \ldots\ldots, \gamma_{n-2}$ and interval $[a_2, b_2], \ldots\ldots$, $[a_{n-1}, b_{n-1}]$, where

I. γ_i are continuous and strictly increasing functions on $[\omega(\overline{\theta}),\overline{\theta}]$
II. $\gamma_i[\omega(\overline{\theta}),\overline{\theta}] \to [a_i,b_i], i = 2,\ldots\ldots,n-1$
III. $a_{i+1} < a_i < b_i$, $i = 2,\ldots,n-2$;
IV. $a_{n-1} > \theta_s$
V. $v\eta(\gamma_{i+1}(y),\gamma_i(y)) - \gamma_i(y) = \delta[v\eta(\gamma_i(y),\gamma_{I-1}(y)) - \gamma_i(y)]$,
$\quad i = 1,2,\ldots n-2$(A14)

Then, it follows that for all $y \in [\omega(\overline{\theta}),\overline{\theta}] : \gamma_{n-1}(y) > \beta(\gamma_{n-2}(y)) > \alpha(\gamma_{n-2}(y))$, which implies that

$$(v-1)\gamma_{n-1}(y) > \delta[v\eta(\gamma_{n-i}(y),\gamma_{n-2}(y)) - \gamma_{n-i}(y)]. \tag{A15}$$

There are two possibilities:
either B.I $\quad v\eta(\underline{\theta},\gamma_{n-i}(y)) - \gamma_{n-i}(y) \geq \delta[v\eta(\gamma_{n-i}(y),\gamma_{n-2}(y)) - \gamma_{n-i}(y)]$,
or B.II $\quad \exists\gamma_n(y) < \gamma_{n-1}(y)$ such that (A15) holds for $i = n-1$.
In case B.I it is easy to show that $\exists\ \tilde{y} \in [\omega(\overline{\theta}),\overline{\theta}]$ such that

$$v\eta(\underline{\theta},\gamma_{n-i}(\tilde{y})) - \gamma_{n-1}(\tilde{y}) \geq \delta[v\eta(\gamma_{n-i}(\tilde{y}),\gamma_{n-2}(\tilde{y})) - \gamma_{n-i}(\tilde{y})].$$

Further for all $y \in [\tilde{y},\overline{\theta}]$ there exists $\gamma_n(y) < \gamma_{n-1}(y)$ such that (A14) holds for $i = n-1$. By definition of $\beta(\overline{\theta})$ and using (A14) it is easy to check that $\gamma_n(\overline{\theta}) = \gamma_{n-1}(\beta(\overline{\theta})) \in [a_{n-1},b_{n-1}]$. Therefore, $\gamma_n(\tilde{y},\overline{\theta}) \to [a_n,b_n]$, $b_n > a_{n-1} > \theta_s$ and $a_n = \underline{\theta}$. In this case set $K = n$, $a_1 = \tilde{y}, b_1 = \overline{\theta}$.

In case B.II, using (A14), one can show that $\gamma_n(y) = (1-\delta)(2-v)/v\gamma_{n-1}(y) + \delta\gamma_{n-2}(y)$. So, γ_n is continuous and increasing in y and y_n : $[\omega(\overline{\theta}),\overline{\theta}] \to [a_n,b_n]$. As $\gamma_n(\overline{\theta}) = \gamma_{n-1}(\beta(\overline{\theta})) > a_{n-1}$ and $\gamma_n(\overline{\theta}) = b_n < \gamma_{n-1}(\overline{\theta}) = b_{n-1}$, it follows that $a_n < a_{n-1} < b_n < b_{n-1}$.

Next, we claim that K, as defined by the above construction, must be finite. For otherwise, there exists an infinite sequence $\{\gamma_i(y)\}_{i=0}^{\infty}$ satisfying (A14) where $\gamma_i(y) < \gamma_{i-1}(y)$ for all i. As $\{\gamma_i(y)\}_{i=0}^{\infty}$ is a monotonic bounded sequence, it converges to some $\gamma^* \geq \underline{\theta} \geq 0$. Taking limits as $i \to \infty$ on both sides of (A15) yields $(v-1)\gamma^* = \delta(v-1)\gamma^*$, a contradiction. Hence, $K < \infty$.

Part V of the statement of the lemma follows from considering

$$\gamma_{i+1}(y) - \gamma_i(y) = \left[(1-\delta)\frac{2-v}{v} - 1\right]\gamma_i(y) + \delta\gamma_{i-1}(y).$$

Defining $\Delta = \gamma_i(y) - \gamma_{i+1}(y)$, it is easy to see that Δ is increasing in $\gamma_i(y)$. As $\gamma_i(y) \geq \omega(\gamma_{i-1}(y))$, it follows that

$$\Delta \geq \left\{\left[1 - \frac{(1-\delta)(2-v)}{v}\right]\left[\frac{\frac{1}{2}\delta v}{2(v-1)+\delta(2-v)} + \frac{(2-v)+2\delta(v-1)}{2v}\right] + \delta\right\}\gamma_{i-1}(y)$$

$$= \left\{-\frac{1}{2}\delta + \left[\frac{[(1-\delta)(2-v)-v][(2-v)+2\delta(v-1)]}{2v^2}\right] + \delta\right\}\gamma_{i-1}(y),$$

which, after some steps, simplifies to

$$\frac{(2-v)(v-1)(1-\delta)^2}{v^2}\gamma_{i-1}(y) > \frac{(2-v)(v-1)(1-\delta)^2}{v^2}\underline{\theta} \equiv \Delta > 0. \qquad \blacksquare$$

The next lemma uses the result derived for the pseudo-model in lemma 3 and shows that *in our true model* (with entry every period), for T large enough and for any initial point y lying in an appropriately chosen interval of qualities close to the highest quality, we can define K marginal qualities (this is the same K as in lemma 3, corresponding to the K last periods of T-period cycle) and these marginal qualities exhibit similar continuity and monotonicity properties as in lemma 3. When T is large enough, for each y the marginal qualities of the true model $\gamma_i^T(y)$ are close to $\gamma_i(y)$, the marginal qualities in the pseudo model, and the corresponding surpluses earned by the sellers of marginal quality are close to that in the pseudo-model. The main reason behind this is that, as shown in lemma 3, the minimum distance between the marginal qualities of the pseudo-model (the minimum size of the interval of quality traded) over all possible y is bounded away from zero. Again, as in lemma 3, we show that the backward process of marginal qualities $\gamma_i^T(y), i = 1, \ldots K$, are properly defined (statement IV of the lemma says that the indifference equation is satisfied; statement V tells us that surpluses are positive) and that each marginal quality is a continuously increasing function of the initial point y (statement III). It is also shown that the Kth marginal quality $\gamma_K^T(y)$ spans an interval around θ_s (statement II), and the surplus that this marginal quality makes lies within a certain interval (statement VI). The latter two properties are important when we want to connect the forward and the backward sequences.

LEMMA 4. *For K as determined by lemma 3 and T large enough, there exist intervals* $[A_i, B_i]$, $i = 1, \ldots K$ *and functions* $\gamma_i^T(y)$, $i = 1,..,K$, $\gamma_0^T(y) = \overline{\theta}, \gamma_1^T(y) = y$, $\gamma_i^T(y) : [A_1, B_1] \to [A_i, B_i]$, $i = 1,..,K$, *such that*

(*I*) $A_{i+1} < A_i < B_{i+1} < B_i$, $i = 1,..,K-1$;
(*II*) $B_1 < \overline{\theta}$, $A_K < \theta_s$, $A_{K-1} > \theta_s$;
(*III*) $\gamma_i^T(y)$ *is continuously differentiable and* $\gamma_i^{T\prime}(y) > 0$, $y \in [A_1, B_1]$, $i = 1,..,K$;
(*IV*) $v\eta^{T-i}(\gamma_{i+1}^T(y),\gamma_i^T(y)) - \gamma_1^T(y) = \delta[v\eta^{T-1+1}(\gamma_i^T(y),\gamma_{i-1}^T(y)) - \gamma_i^T(y)]$,
$i = 1,2,..,K-1$, $y \in [A_1, B_1]$
(*V*) $v\eta^{T-i}(\gamma_{i+1}^T(y),\gamma_i^T(y)) - \gamma_1^T(y) > 0$, $i = 0,1,2,..K-1$;
(*VI*) $(v-1)\gamma_K^T(y) < v\eta^{T-K}(\gamma_K^T(y),\gamma_{K-1}^T(y)) - \gamma_K^T(y) < (1/\delta)\left[(v-1)\gamma_K^T(y) -(v\Delta/2)\right]$.

Proof. Let the functions $\gamma_i(y)$ and intervals $[a_i, b_i]$, $i = 1,..,K$ be as defined by lemma 3 (satisfying I – V). Fix a positive value of u, small enough, such that

$$a_{i+1} < a_i < \gamma_{i+1}(b_1 - u) < \gamma_i(b_i - u). \tag{A16}$$

Define $\hat{a}_i = a_{i,i} = \gamma_1(b_1 - u)$, $i = 1,..,K$. Observe that the γ_is are continuously increasing functions mapping $[\hat{a}_{1,1}]$ to $[\hat{a}_{i,i}]$, $i > 1$. From (A16) it follows that $\hat{a}_{i+1} < \hat{a}_i < \hat{b}_{i+1} < \hat{b}_i$ and $\hat{a}_K \leq \theta_s < \hat{a}_{K-1}$.

For any fixed $h > 0$, let $S(h) = \{(x,y)\colon \underline{\theta} \leq x \leq y \leq \overline{\theta},\ y - x \geq h/2\}$. It follows that $\eta^t(x,y) \to \eta(x,y)$ uniformly on $S(h)$ as $t \to \infty$. Moreover, for any $\rho > 0$ and $K > 1$, $\exists\ \hat{T}(\rho)$ such that for all $t \geq \hat{T}(\rho) + K$:

$$0 \leq \eta(x,y) - \eta^t(x,y) < \rho, \qquad \forall (x,y) \in S(h).$$

Define: $\xi_0(\rho) = 0$,

$$\xi_1(\rho) = 2\rho,$$

$$\xi_n(\rho) = [(1+\delta) + (\frac{2}{v})(1-\delta)]\xi_{n-1}(\rho) + \xi_{n-2}(\rho) + 2\rho \text{ for } n = 2,..,K.$$

Check that for any $n = 1,\ldots,K$, $\xi_n(\rho) \to 0$ as $\rho \to 0$. Further, there exists constant $M > 0$ such that $\xi_n(\rho) < M\rho$, $n = 1,..,K$.

In what follows, we shall impose restrictions on how small ρ should be. Choose $T \geq \hat{T}(\rho) + K$ and any $y \in [\hat{a}_1,\hat{b}_1]$. Set $\gamma_0^T(y) = \overline{\theta}, \gamma_1^T(y) = y$. Using (A13), note that $[v\eta(\gamma_2(y) - \xi_1(\rho),y) - y] - \delta\left[v\eta(y,\overline{\theta}) - y\right] = -v\xi_1(\rho)/2$ so that (for a fixed u)

$$\begin{aligned}
&\left[v\eta^{T-1}(\gamma_2(y) - \xi_1(\rho),y) - y\right] - \delta\left[v\eta^{T-1}(y,\overline{\theta}) - y\right] \\
&= \left[\{v\eta(\gamma_2(y) - \xi_1(\rho),y) - y\} - \delta\{v\eta(y,\overline{\theta}) - y\}\right] \\
&\quad + v\left[\eta^{T-1}(\gamma_2(y) - \xi_1(\rho),y) - \eta(\gamma_2(y) - \xi_1(\rho),y)\right] + \delta v\left[\eta(y,\overline{\theta}) - \eta^T(y,\overline{\theta})\right] \\
&\qquad < -\frac{v\xi_1(\rho)}{2} + \delta\rho v = -v(1-\delta)\rho < 0.
\end{aligned} \tag{A17}$$

Similarly, it can be shown that

$$\left[v\eta^{T-1}(\gamma_2(y)+\xi_1(\rho),y)-y\right]-\delta\left[v\eta^T(y,\bar{\theta})-y\right]>\frac{v\xi_1(\rho)}{2}-v\rho=0. \tag{A18}$$

From (A17) and (A18) and the continuity of the conditional expectation η^t, η^{t-1}, it follows that there exists $\gamma_2^T(y)$, where

$$\begin{aligned}&\gamma_2(y)-\xi_1(\rho)\le\gamma_2^T(y)\le\gamma_2(y)+\xi_1(\rho) \text{ and}\\ &v\eta^{T-1}(\gamma_2^T(y),y)-y=\delta[v\eta^T(y,\bar{\theta})-y].\end{aligned} \tag{A19}$$

Also observe that by definition of $\hat{b}_1\equiv\bar{\theta}-u$, and for ρ small enough,

$$v\eta^T(y,\bar{\theta})-\bar{\theta}>-v\rho+\left[v\eta(\hat{a}_1,\bar{\theta})-\bar{\theta}\right]>-v\rho+v\rho=0.$$

It is easy to check that $\gamma_2^T(y)$ is a continuously differentiable function on $[\hat{a}_1,\hat{b}_1]$. Using the expression for η^T given in (4), differentiating through both sides of (A19) we obtain

$$\begin{aligned}&\left(\frac{v}{2}\right)\left[1-\frac{(T-1)(y-\underline{\theta})^2}{\left((T-1)y-(T-2)\gamma_2^T(y)-\underline{\theta}\right)^2}\right]\gamma_2^{T\prime}(y)=(1-\delta)\left(1-\frac{v}{2}\right)\\ &\quad-\frac{v}{2}\left[\frac{(T-2)\left(\gamma_2^T(y)-\underline{\theta}\right)^2}{\left((T-1)y-(T-2)\gamma_2^T(y)-\underline{\theta}\right)^2}-\delta\frac{T(\bar{\theta}-\underline{\theta})^2}{\left(T\bar{\theta}-(T-1)y-\underline{\theta}\right)^2}\right].\end{aligned} \tag{A20}$$

Observe that

$$\begin{aligned}\frac{(T-1)(y-\underline{\theta})^2}{\left((T-1)y-(T-2)\gamma_2^T(y)-\underline{\theta}\right)^2}&\le\frac{(T-1)(\bar{\theta}-\underline{\theta})^2}{\left((T-2)(y-\gamma_2(y))-\bar{\theta}-\underline{\theta}-(T-2)\epsilon_1(\rho)\right)^2}\\ &\le\frac{(T-1)(\bar{\theta}-\underline{\theta})^2}{\left((T-2)\hat{h}(\epsilon)-\bar{\theta}-\underline{\theta}-(T-2)\epsilon_1(\rho)\right)^2}.\end{aligned}$$

The last expression goes to zero (uniformly in $y\in[\hat{a}_1,\hat{b}_1]$) as $T\to\infty$ and for ρ small enough. Similarly, it can be shown that for ρ small enough

$$\frac{(T-2)\left(\gamma_2^T(y)-\underline{\theta}\right)^2}{\left((T-1)y-(T-2)\gamma_2^T(y)-\underline{\theta}\right)^2}\to 0$$

(uniformly in y) as $T\to\infty$. Hence, for T large enough and ρ small enough, (A20) implies that $\gamma_2^{T\prime}(y)>0$ for all $y\in[\hat{a}_1,\hat{b}_1]$.

Suppose that we have defined continuously differentiable functions $\gamma_i^T(y),\gamma_i^{T\prime}(y)>0$, $y\in[\hat{a}_1,\hat{b}_1]$, $i=2,..,n-1$ such that

$$\gamma_i(y) - \xi_{n-i}(\rho) \leq \gamma_i^T(y) \leq \gamma_i(y) + \xi_{n-i}(\rho) \tag{A21}$$

$$\begin{aligned} &v\eta^{T-i}(\gamma_{i+1}^T(y), \gamma_i^T(y)) - \gamma_i^T(y) \\ &= \delta\big[v\eta^{T-i+1}(\gamma_i^T(y), \gamma_{i-1}^T(y)) - \gamma_i^T(y)\big], \quad i = 1, 2, .., n-2. \end{aligned} \tag{A22}$$

Then, observe that

$$\begin{aligned} &v\eta(\gamma_n(y), \gamma_{n-1}^T(y)) - \gamma_{n-1}^T(y) - \delta\big[v\eta(\gamma_{n-1}^T(y), \gamma_{n-2}^T(y)) - \gamma_{n-1}^T(y)\big] \\ &< v\eta(\gamma_n(y), \gamma_{n-1}^T(y) + \xi_{n-2}(\rho)) - (1-\delta)(\gamma_{n-1}(y) - \xi_{n-2}(\rho)) \\ &- \delta v\eta(\gamma_{n-1}(y) - \xi_{n-2}(\rho), \gamma_{n-2}(y) - \xi_{n-3}(\rho)) \\ &\qquad = \left[\frac{v}{2}(1+\delta) + (1-\delta)\right]\xi_{n-2}(\rho) + \frac{\delta v}{2}\xi_{n-3}(\rho). \end{aligned} \tag{A23}$$

As $\gamma_{i-1}(y) - \gamma_i(y) > \Delta > 0$, there exists ρ small enough such that $\gamma_{i-1}^T(y) - \gamma_i^T(y) > \Delta > 0$. Using (A23) and later, the definition of $\xi_{n-1}(\rho)$, we get

$$\begin{aligned} &[v\eta^{T-n}(\gamma_n(y) - \xi_{n-1}(\rho), \gamma_{n-1}^T(y)) - \gamma_{n-1}^T(y)] - \delta[v\eta^{T-n+1}(\gamma_{n-1}(y), \gamma_{n-2}^T(y)) - \gamma_{n-1}^T(y)] \\ &= [v\eta(\gamma_n(y) - \xi_{n-1}(\rho), \gamma_{n-1}^T(y)) - \gamma_{n-1}^T(y)] - \delta[v\eta(\gamma_{n-1}(y), \gamma_{n-2}^T(y)) - \gamma_{n-1}^T(y)] \\ &\quad + v[\eta^{T-n}(\gamma_n(y) - \xi_{n-1}(\rho), \gamma_{n-1}^T(y)) - \eta(\gamma_n(y) - \xi_{n-1}(\rho), \gamma_{n-1}^T(y))] \\ &\quad - \delta v[\eta^{T-n+1}(\gamma_{n-1}(y), \gamma_{n-2}^T(y)) - \eta(\gamma_{n-1}(y), \gamma_{n-2}^T(y))] \\ &\quad < \left[\frac{v}{2}(1+\delta) + (1-\delta)\right]\xi_{n-2}(\rho) + \frac{\delta v}{2}\xi_{n-3}(\rho) + \delta v\rho - \frac{v\xi_{n-1}(\rho)}{2} \\ &\qquad = \frac{v}{2}\left[\{(1+\delta) + \frac{2}{v}(1-\delta)\}\xi_{n-2}(\rho) + \delta\xi_{n-3}(\rho) + 2\delta\rho - \xi_{n-1}(\rho)\right] < 0. \end{aligned}$$

Similarly, it can be shown that

$$\begin{aligned} &v\eta(\gamma_n(y), \gamma_{n-1}^T(y)) - \gamma_{n-1}^T(y) - \delta[v\eta(\gamma_{n-1}(y), \gamma_{n-2}^T(y)) - \gamma_{n-1}^T(y)] \\ &\qquad > -\left(\frac{v}{2}\right)[\{(1+\delta) + \frac{2}{v}(1-\delta)\}\xi_{n-2}(\rho) + \delta\xi_{n-3}(\rho)], \end{aligned}$$

so that

$$\begin{aligned} &v\eta^{T-n}(\gamma_n(y) + \xi_{n-1}(\rho), \gamma_{n-1}^T(y)) - \gamma_{n-1}^T(y) - \delta[v\eta^{T-n+1}(\gamma_{n-1}(y), \gamma_{n-2}^T(y)) - \gamma_{n-1}^T(y)] \\ &\qquad > -(\frac{v}{2})\left[\{(1+\delta) + \frac{2}{v}(1-\delta)\}\xi_{n-2}(\rho) + \delta\xi_{n-3}(\rho) + 2\rho - \xi_{n-1}(\rho)\right] = 0. \end{aligned}$$

Thus, there exists $\gamma_n^T(y)$ such that

$$\gamma_n(y) - \xi_{n-1}(\rho) \leq \gamma_2^T(y) \leq \gamma_2(y) + \xi_{n-1}(\rho) \tag{A24}$$

$$v\eta^{T-n}(\gamma_n^T(y), \gamma_{n-1}^T(y)) - \gamma_{n-1}^T(y)$$
$$= \delta[v\eta^{T-n+1}(\gamma_{n-1}^T(y), \gamma_{n-2}^T(y)) - \gamma_{n-1}^T(y)]. \qquad \text{(A25)}$$

Also, observe that

$$\begin{aligned}
&v\eta^{T-n}(\gamma_n^T(y),\gamma_{n-1}^T(y)) - \gamma_{n-1}^T(y)\\
&= \delta[v\eta^{T-n+1}(\gamma_{n-1}^T(y),\gamma_{n-2}^T(y)) - \gamma_{n-1}^T(y)]\\
&\geq \delta[v\eta(\gamma_{n-1}^T(y),\gamma_{n-2}^T(y)) - \gamma_{n-1}^T(y) - v\rho]\\
&\geq \delta[(v-1)\gamma_{n-1}^T(y) - v\rho] \geq \delta[(v-1)\underline{\theta} - v\rho] > 0.
\end{aligned}$$

From (A25), and the exact functional forms of η^T, it follows that $\gamma_n^T(y)$ is continuously differentiable on $[\hat{a}_1,\hat{b}_1]$. Differentiating through (A25) yields

$$\frac{v}{2}\left[1 - \frac{(T-n)(\gamma_{n-1}^T(y) - \underline{\theta})^2}{((T-n)\gamma_{n-1}^T(y) - (T-n-1)\gamma_n^T(y) - \underline{\theta})^2}\right]\gamma_n^{T\prime}(y) = (1-\delta)\left(1 - \frac{v}{2}\right)\gamma_{n-1}^{T}{}'(y)$$

$$-\frac{v}{2}\left[\frac{(T-n-1)(\gamma_n^T(y) - \underline{\theta})^2}{((T-n)\gamma_{n-1}^T(y) - (T-n-1)\gamma_n^T(y) - \underline{\theta})^2} - \delta\cdot \frac{(T-n+1)\left(\gamma_{n-2}^T(y) - \underline{\theta}\right)^2}{\left((T-n+1)\gamma_{n-2}^T(y) - (T-n)\gamma_{n-1}^T(y) - \underline{\theta}\right)^2}\right].$$

$$\gamma_{n-1}^{T}{}'(y) + \delta\frac{v}{2}\left[1 + \frac{(T-n)\left(\gamma_{n-1}^T(y) - \underline{\theta}\right)^2}{\left((T-n+1)\gamma_{n-2}^T(y) - (T-n)\gamma_{n-1}^T(y) - \underline{\theta}\right)^2}\right]\cdot \gamma_{n-2}^{T}{}'(y). \qquad \text{(A26)}$$

Using the fact that

$$\gamma_{n-1}^T(y) - \gamma_n^T(y) \geq \gamma_{n-1}(y) - \epsilon_{n-2}(\rho) - \gamma_n(y) - \epsilon_{n-1}(\rho) \geq \Delta - (\epsilon_{n-2}(\rho) + \epsilon_{n-1}(\rho))$$

and $\gamma_{n-2}^T(y) - \gamma_{n-1}^T(y) \geq \Delta + (\epsilon_{n-3}(\rho) + \epsilon_{n-2}(\rho))$,
it can be shown that for ρ small enough, as $T \to \infty$:

- the term in square brackets on the left-hand side of (A26) converges to 1 uniformly in $y \in [\hat{a}_1,\hat{b}_1]$;
- the term in square brackets multiplying $\gamma_{n-1}^{T}{}'(y)$ on the right-hand side of (A26) converges to 0 uniformly in $y \in [\hat{a}_1,\hat{b}_1]$.

Hence, there exists $\rho > 0$ small enough and T large enough such that for all $y \in [\hat{a}_1,\hat{b}_1]$, $\gamma_n^{T\prime}(y) > 0$, and, by induction, there exists $\rho > 0$ and T large enough such that for all $T \geq \tau$, $\gamma_n^T(y)$ is strictly increasing in y on $[\hat{a}_1,\hat{b}_1]$, $n = 1, 2, \ldots, K$.

As $\gamma_i^T(\hat{\boldsymbol{a}}_1) \in [\gamma_i(\hat{\boldsymbol{a}}_1) - \xi_i(\rho), \gamma_i(\hat{\boldsymbol{a}}_1) + \xi_i(\rho)] = [\hat{\boldsymbol{a}}_i - \xi_i(\rho), \hat{\boldsymbol{a}}_i + \xi_i(\rho)]$ and $\gamma_i^T(\hat{\boldsymbol{b}}_1) \in [\gamma_i(\hat{\boldsymbol{b}}_1) - \xi_i(\rho), \gamma_i(\hat{\boldsymbol{b}}_1) + \xi_i(\rho)] = [\hat{\boldsymbol{b}}_i - \xi_i(\rho), \hat{\boldsymbol{b}}_i + \xi_i(\rho)]$, we can choose ρ small enough so that

$$\gamma_{i+1}^T(\hat{a}_1) < \gamma_i^T(\hat{a}_1) < \gamma_{i+1}^T(\hat{b}_1) < \gamma_i^T(\hat{b}_1).$$

As γ_i^Ts are continuous, let $[A_i, B_i]$ denote the image $[\hat{a}_{1,1}]$ under the mapping γ_i^T. Then,

$$A_{i+1} < A_i < B_{i+1} < B_i, \qquad i = 1,..,K-1.$$

Using the definition of K from the previous lemma, for ρ small enough, $A_K < \theta_s$.

To prove VI, define $s_K \equiv v\eta^{T-K}(\gamma_K^T(y),\gamma_{K-1}^T(y)) - \gamma_K^T(y)$ as the surplus of $\gamma_K^T(y)$ in the period where $\gamma_{K-1}^T(y)$ is the highest quality traded. To prove the first inequality of VI consider $s_K - (v-1)\gamma_K^T(y)$. For large enough T and using V of lemma 3, it is easily seen that this expression is strictly larger than 0.

To prove the second inequality of VI, we distinguish two cases:

(*i*) there exists a $\gamma_{K+1}^T(y) \geq \underline{\theta}$ such that

$$v\eta^{T-K+1}(\gamma_{K+1}^T(y),\gamma_K^T(y)) - \gamma_K^T(y) = \delta s_K^T(y), \text{ or}$$

(*ii*) $v\eta^{T-K+1}(\underline{\theta},\gamma_K^T(y)) - \gamma_K^T(y) > \delta s_K^T(y)$.

In the first case (the second case works similarly and won't be treated explicitly), we consider $v\eta(\gamma_{K+1}^T(y),\gamma_K^T(y)) - \gamma_K^T(y) - [(v-1)\gamma_K^T(y) - v\Delta/2]$. For large enough T and using V of lemma 3, this expression is strictly smaller 0. ■

Finally, we can put all the elements of the proof together. From lemma 4 we know that there exists a T^* such that for all $T > T^*$ in the backward sequence surpluses of marginal sellers are continuous functions of the starting point y. Moreover, for all $T > T^*$ all marginal qualities $\gamma_i^T(y)$ are increasing functions and therefore invertible. This allows us to rewrite surpluses as functions of the marginal sellers themselves: $s_K^T(\theta)$. As $\gamma_K^T(y)$ covers an interval around θ_s, and as the forward sequence $\theta_t(\theta_1)$ with $t = T-k$ covers a subset of this interval, we can actually redefine the surpluses from the downward sequence in terms of a continuous function of $\theta_1 : s_K^T(\theta_1)$. Using the fact that this surplus is within a certain interval, we know that for any θ_1 in the relevant interval $[\hat{\theta}_1(T),\theta_s]$, the surplus $s_K^T(\theta_1)$ constructed in this way is strictly positive and smaller than $[(v-1)\theta - v\Delta/2]/\delta$. The above argument holds true for any T larger than some T^*. On the other hand, from the forward process we know that on the same interval surpluses are a continuous function ranging from 0 (when $\theta_1 = \theta_s$) to a number close to $(v-1)\theta/\delta$. More particularly, lemma 2 implies that we can choose any particular value of the surplus (or in terms of lemma 2: a particular value of ϵ) for the forward process larger than $[(v-1)\theta - v\Delta/2]/\delta$ such that there exists a particular value of θ_1 and a t such that this surplus is reached within t periods starting from this particular value of θ_1. If this value of t is larger than T^*, we can appeal to the intermediate value theorem to show that there must be a θ_1 where the surpluses of the above constructed processes and the forward process are identical. As the forward and the backward sequences satisfy the necessary equilibrium properties, it follows that starting from this particular θ_1 there is an equilibrium cycle up to $\overline{\theta}$. If this value of t is smaller than T^*, we can choose a small enough value of ϵ (a surplus close enough to $(v-1)\theta/\delta$) in such a way that the value of t needed to reach this surplus is larger than T^*.

Choose an $\epsilon > 0$ small enough and consider the corresponding large T and the functions $s_T(\theta_1)$ and $\theta_T(\theta_1)$ as defined in lemma 1 and $\hat{\theta}_1(T)$ as defined in lemma 2. It follows that for a given $\theta_1 = \hat{\theta}_1(T)$ there either exists θ_{T+1} such that $v\eta^T(\theta_T,\theta_{T+1}) - \theta_T$ is larger than $[(v-1)\theta_T(\theta_1)] - \epsilon/\delta$ or $\delta[v\eta(\theta_T,\overline{\theta}) - \theta_T] < s_T(\theta_T(\hat{\theta}_1(T))$. In the last case, an intermediate value argument can be used to show there is a T-period dynamic equilibrium reaching $\overline{\theta}$. To see this note that at $\theta_1 = \theta_s$ $\delta[v\eta(\theta_T,\overline{\theta}) - \theta_T] > s_T(\theta_T) = 0$. As $s_T(\theta_1)$ and $\theta_T(\theta_1)$ are continuous functions, there must be a θ_1 with $\hat{\theta}_1(T) < \theta_1 < \theta_s$ such that $\delta[v\eta(\theta_T,\overline{\theta}) - \theta_T] = s_T(\theta_T)$. So, we can concentrate on the first case. Moreover, $\theta_s \leq \theta_T(\theta_1) < \theta_s + \tilde{\epsilon}$ for all $\theta_s \geq \theta_1 \geq \hat{\theta}_1(T)$.

Recall from lemma 4 that we have defined functions $\gamma_i^T(y)$ and $\eta^{T-i}(\gamma_i^T(y),\gamma_{i-1}^T(y))$ on the interval $[A_1, B_1]$, $i = 1,\ldots,K$. As $\gamma_i^{T\prime}(y) > 0$ (see lemma 4), the functions $\gamma_i^T(y)$ are invertible, $i = 1,\ldots K$, on the interval $[\hat{a}_1,\hat{b}_1]$. Hence, there exists a continuous increasing function $g{:}[A_K,B_K] \to [\hat{a}_1,\hat{b}_1]$ such that $\gamma_K^T(g(\theta)) = \theta$ for all $\theta \in [A_K,B_K]$. This allows us to redefine $s_K^T \equiv v\eta^{T-K}(\gamma_K^T(y),\gamma_{K-1}^T(y)) - \gamma_K^T(y)$ as being defined on $[A_K,B_K] : s_K^T(\theta) \equiv v\eta^{T-K}(\theta,\gamma_{K-1}^T(g(\theta))) - \theta$. It follows that $s_K^T(\theta)$ is a continuous function on $[A_K,B_K]$. Furthermore, $\frac{1}{\delta}[(v-1)\theta - v\Delta/2] > s_K^T(\theta) > (v-1)\theta > 0$ for all $\theta \in [A_K,B_K]$. As we can choose $\tilde{\epsilon}$ such that $A_K < \theta_s < \theta_s + \tilde{\epsilon} < B_K$, we can use the function $\theta_T(\theta_1)$ to write $s_K^T(\theta_T(\theta_1))$ as a continuous function defined on $\theta_s \geq \theta_1 \geq \hat{\theta}_1(T)$.

So, at $\theta_1 = \theta_s$, $s_T = v\eta^T(\theta_T,\theta_{T+1}) - \theta_T = 0 < s_K^T(\theta_s)$. At $\theta_1 = \hat{\theta}_1(T)$, we know that $v\eta^T(\theta_{T-K},\theta_{T-K+1}) - \theta_{T-K}$ is larger than $[(v-1)\theta_{T-K}(\theta_1) - \epsilon]/\delta$, which in turn is larger than $s_K^T(\theta_{T-K}(\theta_1))$, since ϵ is close to 0 for large T. As $s_T(\theta_1)$, $\theta_T(\theta_1)$ and $s_K^T(\theta_T(\theta_1))$ are continuous functions, there must be a θ_1 such that $s_T(\theta_t) = s_K^T(\theta_T(\theta_1))$. ■

Appendix B: Proof of other propositions

Proof of proposition 4. It is clear that repeating the static equilibrium is an equilibrium in the dynamic model and that this equilibrium is weakly monotonic. We will first show that there does not exist another equilibrium sequence $\{\theta_t\}_{t=1}^{\infty}$ with $\theta_t = \theta_{t-1}$ for some t. Suppose that there was such a sequence and that τ is the first period in which $\theta_\tau = \theta_{\tau-1}$. A first implication is that $\theta_{\tau-1} \leq \theta_s$, since otherwise $s_\tau < 0$. A second implication is that $p_\tau \leq p_{\tau-1}$, and hence, that

$s_\tau \le s_{\tau-1}$. There are several possibilities as regards $\theta_{\tau-1}$ and τ: $\tau = 2$ and $\theta_{\tau-1} < \theta_s$, $\theta_{\tau-1} = \underline{\theta}$ or $\theta_{\tau-1} > \theta_{\tau-2}$. In all cases, $s_{\tau-1} > 0$. This in turn means that there must be a $k \ge 1$ such that $s_{\tau-1} = \delta^{k+1}\,(p_{\tau+k} - \theta_{\tau-1})$, and hence, that $s_\tau \le \delta^{k+1}(p_{\tau+k} - \theta_{\tau-1}) < \delta^k(p_{\tau+k} - \theta_\tau)$. This implies that θ_τ prefers to wait until period $\tau + k$ before selling – a contradiction.

We will next prove that there neither exists an equilibrium sequence $\{\theta_t\}_{t=1}^{\infty}$ with $\theta_t > \theta_{t-1}$. The proof is by contradiction. Suppose that there exists such a sequence. Then, it must be the case that there exists a $\tilde{\theta} \le \overline{\theta}$ such that $\lim_{t\to\infty} \theta_t = \tilde{\theta}$. We will distinguish two cases: (a) $\tilde{\theta} > \theta_s$ and (b) $\tilde{\theta} = \theta_s$. (It is clear that in equilibrium it cannot be the case that $\tilde{\theta} < \theta_s$, since there will be sellers $\theta > \tilde{\theta}$ who prefer strictly to sell, but will never do so according to the stipulated equilibrium).

In the first case, we can take a (small) fixed $\epsilon > 0$ and a large T such that $\theta_t > \theta_s + \epsilon$ for all $t > T$. Equation (A4) in lemma 2 expresses θ_t as a function of θ_{t-1} and s_{t-1}. It is clear that for a given θ_{t-1}, θ_t is an increasing function of s_{t-1}. Combining these two facts and substituting $s_{t-1} = 0$ in (A4) gives

$$\theta_t - \theta_{t-1} > \left(\frac{v-1}{v}\right)\theta_{t-1}\left[-1 + \sqrt{1 + \frac{v(2-v)(\theta_{t-1} - \underline{\theta})(\theta_{t-1} - \theta_s)}{(v-1)^2 t\theta_{t-1}^2}}\right].$$

For large enough t, the expression $[v(2-v)(\theta_{t-1} - \underline{\theta})(\theta_{t-1} - \theta_s)]/[(v-1)^2 t\theta_{t-1}^2]$ is strictly smaller than 8. As $\sqrt{1+x} > 1 + x/4$ for all $x \in (0,8)$, this expression is larger than $[(2-v)(\theta_{t-1} - \underline{\theta})(\theta_{t-1} - \theta_s)]/[4(v-1)t\theta_{t-1}]$. As there exist a (small) fixed $\epsilon > 0$ and a large T such that for all $t > T$, $\theta_t > \theta_s + \epsilon$, this expression is larger than A/t, for some constant $A > 0$.

Hence, for all $t > T$, we can write $\theta_t = \theta_T + \Sigma_{k=T}^{t}(\theta_{k+1} - \theta_k) > \theta_T + \Sigma_{k=T}^{t} A/k$. As $\lim_{t\to\infty} \Sigma_{k=T}^{t} A/k = \infty$, this implies that $\lim_{t\to\infty} \theta_t = \infty$. This contradicts the fact that there exists a $\tilde{\theta} \le \overline{\theta}$ such that $\lim_{t\to\infty} \theta_t = \tilde{\theta}$.

Let us then concentrate on the case $\tilde{\theta} = \theta_s$. In this case for large t, θ_t is close to but strictly smaller than θ_s, $s_t > 0$, and $\lim_{t\to\infty} s_t = 0$. This implies that for any $\epsilon > 0$ there exists a T such that for all $t > T$, $s_t < \epsilon$. Equation (A5) in lemma 2 tells us that for $\theta_t < \theta_s$, $\alpha s_{t+1} > s_t$ if

$$s_t\left[(t+1)\left(\frac{1}{\delta} - \frac{1}{\alpha}\right)\left(\frac{v}{\alpha} - \frac{v-2}{\delta}\right) + \frac{2(\theta_t - \underline{\theta})}{\delta}\right] < 2(v-1)(t+1)\left(\frac{1}{\delta} - \frac{1}{\alpha}\right)\theta_t.$$

This expression can be rewritten as

$$s_t < \frac{2(v-1)\theta_t}{\frac{v}{\alpha} + \frac{2-v}{\delta} + \frac{2(\theta_t - \underline{\theta})}{(t+1)(1-\frac{\delta}{\alpha})}}.$$

For large t, the R.H.S. of this expression is increasing in θ_t. So, we can certainly argue that for large t, $\alpha s_{t+1} > s_t$ if

$$s_t < \frac{2(v-1)\underline{\theta}}{\frac{v}{\alpha} + \frac{2-v}{\delta}}.$$

This contradicts the hypothesis we started with, namely, $\lim_{t\to\infty} s_t = 0$. ■

Proof of proposition 5. The proof is by contradiction. Suppose that for all t, $\theta_t \leq \theta_s$. It is easy to see that it cannot be the case that $\sup_t \theta_t < \theta_s$, as this would imply that there are qualities $\theta > \sup_t \theta_t$ that would make a strictly positive surplus by selling, but they would never sell in such an equilibrium. Also, it cannot be the case that for some t, $\theta_t = \theta_s$. This is because in a non-monotonic equilibrium $\theta_1 < \theta_s$, but this would imply that the marginal seller in the first period t, such that $\theta_t = \theta_s$, makes a strictly positive surplus. Hence, by continuity there would be qualities θ just above θ_s that would make a strictly positive surplus by selling, but they would never sell in such an equilibrium.

We are left with the case that for all t, $\theta_t < \theta_s$ and $\sup_t \theta_t = \theta_s$. In this case there has to be an increasing subsequence $\{\theta_{t+k_h}\}_{h=0}^{\infty}$ with

(i) $\theta_{t+k_1} < \theta_{t+k_2} < \dots$
(ii) $s_{t+k_1} = \delta^{k_2-k_1}(p_{t+k_2} - \theta_{t+k_1})$ and
(iii) $\lim_{h\to\infty} \theta_{t+k_h} = \theta_s$, $\lim_{h\to\infty} p_{t+k_h} = \theta_s$ and $\lim_{h\to\infty} s_{t+k_h} = 0$.

Choose $t + k_0$ to be a large number and let θ_{t+k_0} be close to θ_s. Write $\epsilon_\theta = \theta_s - \theta_{t+k_0}$. Take $t + k_0$ and θ_{t+k_0} as fixed numbers. Define θ^e to be the expected quality in period $t + k_1$ conditional upon $\theta_{t+k_1} = \theta_{t+k_0}$. We will distinguish two cases. First, there is a fixed ϵ, independent of t, with $0 < v\epsilon < (1-\delta)(v-1)\theta_{t+k_0} - (\frac{v}{2} + \delta)\epsilon_\theta$ such that $\theta^e \geq \theta_{t+k_0} - \epsilon$. We will argue that in this case that $\theta_{t+k_2} > \theta_s$. If $\theta^e \geq \theta_{t+k_0} - \epsilon$, then $s_{t+k_1} \geq (v-1)\theta_{t+k_1} - v\epsilon - (1-\frac{v}{2})(\theta_{t+k_1} - \theta_{t+k_0})$, which is strictly larger than $(v-1)\theta_{t+k_1} - v\epsilon - (1-\frac{v}{2})\epsilon_\theta$. Using $s_{t+k_1} = \delta^{k_2-k_1}(p_{t+k_2} - \theta_{t+k_1})$, it follows that

$$s_{t+k_2} > \frac{(v-1)\theta_{t+k_1} - v\epsilon - (1-\frac{v}{2})\epsilon_\theta}{\delta^{k_2-k_1}} - (\theta_{t+k_{12}} - \theta_{t+k_1}).$$

However, since for any t, s_t cannot exceed $(v-1)\theta_t$, this implies that it is necessarily the case that $(v-1+\delta)\theta_{t+k_1} - v\epsilon - (1-v/2)\epsilon_\theta < \delta v \theta_{t+k_2}$. If ϵ satisfies the condition above, then this implies that $\theta_{t+k_2} > \theta_s$.

Second, suppose there exists $0 < \epsilon < [((1-\delta)(v-1)/v)]\theta_{t+k_0} - \delta(\theta_s - \theta_{t+k_0})$ with $\theta^e < \theta_{t+k_0} - \epsilon$. There is a finite $x < k(\theta_{t+k_0} - \underline{\theta}) < k(\theta_s - \underline{\theta}))$ such that we can write

$$s_{t+k_1} = v\frac{x\theta^e + (t+k_1)(\theta_{t+k_1} - \theta_{t+k_0})\frac{\theta_{t+k_1} + \theta_{t+k_0}}{2}}{x + (t+k_1)(\theta_{t+k_1} - \theta_{t+k_0})} - \theta_{t+k_1}.$$

The sign of the derivative of s_{t+k_1} with respect to θ_{t+k_1} is the sign of

$$-\left(1-\frac{v}{2}\right)(t+k)\theta^2_{t+k_1}+(2-v)[(t+k)\theta_{t+k_0}-x]\theta_{t+k_1}+2x\theta_{t+k_0}$$
$$-\left(1-\frac{v}{2}\right)(t+k)\theta^2_{t+k0}-vx\theta^e-\frac{x^2}{t+k}.$$

At $\theta_{t+k_1}=\theta_{t+k_0}$ this expression reduces to $vx(\theta_{t+k_0}-\theta^e)-(x^2/t+k)$, which for large t, is strictly positive in this second case. Hence, we can conclude that in a right neighbourhood of θ_{t+k_0}, s_{t+k_1} is increasing in θ_{t+k_1} and that after reaching a maximum s_{t+k_1} will be decreasing in θ_{t+k_1}. Moreover, at $\theta_{t+k_1}=\theta_{t+k_0}$ we know that $s_{t+k_1}=s_{t+k_0}/\delta^{k_1-k_0}\geq s_{t+k_0}/\delta$ and it is easy to show (see lemma 2 for a similar calculation) that for large $t+k_0$ and a given θ_{t+k_0}, the maximum s_{t+k_1} is close to $(v-1)\theta_{t+k_1}$. Finally, whenever s_{t+k_1} is decreasing in θ_{t+k_1}, it is decreasing at a rate approximately equal to $(1-(v/2))$. All observations together imply that either $s_{t+k_1}\geq s_{t+k_0}/\delta$ or θ_{t+k_1} will be larger than θ_s. Whichever applies, it contradicts requirement (iii) above. ■

Proof of proposition 6. For $\theta^*\in(\theta_s,\bar{\theta}]$, suppose that $\lim\inf_{\delta\uparrow 1}\ \tau(\delta,\ \theta^*)<\infty$. Consider a sequence $\{\delta_n\}\uparrow 1$ such that $\{\tau(\delta_n,\theta^*)\}$ is bounded above, say, by $K<\infty$. Then for each n, there exists an equilibrium where a seller with quality θ^* entering in period 1 sells her good in a period less than K at a price we shall denote by $p(\delta_n)$. Now, for each n, consider the first period of positive trading in such an equilibrium and let $p_1(\delta_n)$ be the price and $\theta_1(\delta_n)$ the highest quality traded in that period. Obviously, $\theta_1(\delta_n)\leq\theta_s$, and the first period in which θ^* is traded must be later than this period. Let τ_n also indicate the time difference between the first period of positive trading and the first period in which θ^* is traded.

The seller with quality $\theta_1(\delta_n)$ prefers to sell in period 1 rather than in period τ_n, and as $\tau_n\leq K$, we have $p_1(\delta_n)-\theta_1(\delta_n)\geq(\delta_n^K)(p(\delta_n)-\theta_1(\delta_n))$. As $p_1(\delta_n)\leq\theta_s$ and as $p(\delta_n)\geq\theta^*$ (because quality θ^* is traded in period τ_n at price $p(\delta_n)$), we have $\theta_s-\theta_1(\delta_n)\geq\delta_n^K(\theta^*-\theta_1(\delta_n))$. As $\theta_1(\delta_n)\in[\underline{\theta},\theta_s]$ for all n and $\{\delta_n\}\uparrow 1$, taking $n\to\infty$ on both sides of the inequality, we have $\theta_s\geq\theta^*$, a contradiction. ■

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