Online Appendix to Dynamic Pricing with Uncertain Capacities

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The following Appendix contains supplementary materials for the paper "Dynamic pricing with Uncertain Capacities". It is organized in the following way. Section A discusses the equilibrium outcome under an alternative tie-break assumption, when firms set the same price. Section B relaxes the assumption that firms are able to observe each others' prices. Section C contains additional materials on the extension of Random Arrival of Consumers in Section 4.3.1 from the main paper. Similarly, Section D expands on Section 4.3.4 about Captive Consumers from the main paper. Finally, Section E displays additional graphs for different model parameters' values.

A Alternative Tie-breaking Rules

In this part, we show that the equilibrium outcome of the game with hidden capacities does not depend on the specific tie-breaking rule that we assumed in the main paper. In particular, we are interested in the alternative tie-breaking rule that allocates the demand evenly among all firms that charge the lowest price instead of randomly allocating all demand to one firm.

First, as mentioned in footnote 11 of the main text observe that the pooling equilibrium outcome can be supported by an asymmetric equilibrium in which one of the firms, say firm 1, charges $p = \alpha$, while its rival randomizes in an interval $(p, p + \varepsilon)$, for $\varepsilon > 0$ sufficiently small. In this case, there are no ties in equilibrium (so the tie-breaking rule - whatever it is - does not apply) and the same equilibrium outcome emerges. In particular, a constrained firm 2 has no incentive to deviate as its expected (second-period) profit equals α and any deviation in the first period cannot yield more profit. Like in Section 4, whether an unconstrained firm 2 has an incentive to deviate depends on the out-of-equilibrium beliefs if firm 1 observe its rival sets a first-period price smaller than α. As only unconstrained firms may have such an incentive to deviate, it is natural that firm 1 believes a deviating, undercutting firm is unconstrained, resulting in zero expected second-period profits, implying that a deviation by an unconstrained firm 2 is not profitable.

Second, the pooling equilibrium in pure strategies can also be supported by any tie-breaking rule. Indeed, consider the pooling equilibrium in which both firms charge α in the first period and then randomize in the second period in $[\alpha, 1]$. Moreover, consider that given the first-period prices firm i's first-period market share equals q_i. Notice that the on-path continuation game following such a split of demand in the first period is such that with probability $1-\alpha$ firm i is unconstrained while with probability α firm i has capacity to serve exactly $1 - q_i$ share of the market. It follows that the continuation profit of a constrained firm is $(1-q_i)\alpha$, and this firm will randomize in the interval $[\alpha, 1]$. Thus, the overall expected pay-off of a constrained firm i is equal to α and independent of q_i. Deviating in either period cannot improve upon this expected profit. The unconstrained firm setting the lowest price in the support sells for sure and obtains α, which is also its maximum continuation profit. Thus, the overall expected pay-off of a constrained firm i equals $\alpha(1 + q_i)$. For the same reason as above, an unconstrained firm also does not want to deviate in the first period. Thus, for any tie-breaking rule the pooling equilibrium remains an equilibrium.

Finally, consider the semi-separating equilibrium strategies described in Section 4.1. If a tie occurs, it must be at the highest equilibrium price $\bar{p}$. If a firm sells quantity q_i at $\bar{p}$, it expects to make $q_i\bar{p} + (1 - q_i)\theta(\bar{p}) = \bar{p}$, regardless of q_i. Hence, the same first-period equilibrium strategies constitute an equilibrium (together with the natural adaptation of the second-period strategy).

B Hidden Prices

In certain markets it may be difficult for a firm to observe its rival's price. Since prices act as signals, it is natural to ask whether the results would significantly differ in an environment in which prices are unobserved. We now argue that the pooling equilibrium outcome we characterized in the beginning of Section 4 is the natural outcome for such markets as it is the unique equilibrium outcome.

First, just like in the case of alternative tie-breaking rules, it is easy to see that the asymmetric equilibrium that induces the pooling equilibrium outcome remains an equilibrium regardless of the information about prices. Second, it is straightforward to check that the symmetric pooling equilibrium breaks down if prices are hidden. As deviations are not observed if they do not result in another firm selling in the first period, an unconstrained firm can undercut its competitor and pretend that it simply was lucky in the first period and was selected by the tie-breaking rule. For this very same reason, a separating equilibrium cannot be supported. Third, a similar argument can be used to show that no semi-separating equilibrium can exist in this case. Recall that in a semi-separating equilibrium, the expected profit of selling the second unit in the second period must be the same, regardless of the price chosen in the first period. This requires that higher first-period prices induce higher continuation profits. But any firm that charges a marginally lower price in the first period can simply mimic the second-period behavior of a firm who charged a higher first-period price, making such a deviation profitable (as it is unobserved).

C Random Arrival of Consumers

In this Appendix we show that not only the pooling equilibrium can be extended to random arrivals of consumers, but also the semi-separating equilibria. Given that in every period a consumer is present only with probability $\lambda \in (0,1)$, the second period equilibrium profit of the highest-pricing firm is $\lambda^2\theta(p)$, where $\theta(p)$ is the posterior probability that it will be a monopolist in the second period. This is also the equilibrium profit of her rival (in case it remains active). In the first period, then, a constrained firm who charges p expects

$$\Pi^c(p) = Q(p)\left(\lambda p + (1-\lambda)\theta(p)\lambda^2\right) + \int_{\underline{p}}^{p} -Q'(x)\lambda^2\theta(x)dx,$$

while the unconstrained firm expects a profit of $\Pi^c + Q(p)\theta(p)\lambda^2$. As in the main model, this implies that $Q(p)\theta(p)$ must be a constant. The first order condition reads:

$$Q'(p)\lambda\left(p - \theta(p)\lambda^2\right) + Q(p)\lambda(1 + (1-\lambda)\lambda^2\theta'(p)) = 0$$

since $Q'(p)\theta(p) + Q(p)\theta'(p) = 0$, we obtain

$$p - \theta(p)\lambda^2(2-\lambda) - \frac{\theta(p)}{\theta'(p)} = 0$$

which yields

$$\theta(p) = \frac{-p}{\lambda^2(2-\lambda)W_{-1}\left(\frac{pc}{\lambda^2(2-\lambda)}\right)}.$$

At the upper bound, we must have that the constrained firm is indifferent between selling or not: $\lambda\bar{p} = \lambda^2\theta(\bar{p})$, so $\theta(\bar{p}) = \bar{p}/\lambda$. The unconstrained firm must also be indifferent. Hence, $2\theta(\bar{p}^-) = \theta(\bar{p}) = \frac{\bar{p}}{\lambda}$. Hence,

$$\lambda^2(2-\lambda)W_{-1}\left(\frac{\bar{p}c}{\lambda^2(2-\lambda)}\right) = \frac{\bar{p}}{2\lambda}.$$

Solving this equation yields

$$c = \frac{1}{2\lambda} + \lambda^2(2-\lambda)\log(2\lambda p).$$

Using the same arguments derived in the Proof of Proposition 4, we obtain that for every $\bar{p} \in [\frac{\lambda\alpha}{2-\lambda}, \lambda]$, there exists a semi-separating (pooling if $\bar{p} = \lambda\alpha$), such that the first period prices lie in $[\frac{\lambda\alpha}{2-\lambda}, \bar{p}]$ and the second-period prices in $[\theta(p)\lambda, 1]$.

D Captive Consumers

Complete Information about Capacities Each firm has the same probability $\sigma \leq \frac{1}{2}$ of capturing the whole demand of the arriving consumer. There are 3 cases: (i) both firms constrained, (ii) one firm constrained, one unconstrained, (iii) both unconstrained.

Case (i): Even in this simple case with two constrained firms, which can split the market, the equilibrium from the perfect competition case does not carry on. In other words, an equilibrium where both firms price at the monopoly price in the first period does not exist. If it did, both firms ex-ante expect $\pi = \frac{1}{2}1 + \frac{1}{2}(1-\sigma) = \frac{2-\sigma}{2}$ as the firm that does not sell today, expects $1-\sigma$ tomorrow. Deviations upwards are not profitable, but a firm deviating downwards expects $\pi^C(1^-) = (1-\sigma)1^- + \sigma(1-\sigma) = 1-\sigma^2 > \frac{2-\sigma}{2}$. In this sense, there also does not exist an equilibrium in pure strategies, where the firms price at a price $p^* < 1$ as the same incentives to undercut arise.

Looking for an equilibrium, where both firms mix implies that the firms have an expected profit of

$$\pi^C(p) = [\sigma + (1-2\sigma)(1-F^C(p))]p + [\sigma + (1-2\sigma)F^C(p)](1-\sigma)$$

The firms need to be indifferent between selling at p or pricing at the monopoly price. This implies that their mixing strategy is given by $F^C(p) = \frac{1-2\sigma+2\sigma^2-(1-\sigma)p}{(1-2\sigma)(1-\sigma-p)}$ with support $p \in [\frac{1-2\sigma+2\sigma^2}{1-\sigma}, 1]$. Both firms' ex-ante expected profit is equal to $\pi^C = 1-\sigma+\sigma^2$.

In the limits of $\sigma \in [0, \frac{1}{2}]$ the lower bound of the price distributions converges to the monopoly price. Hence, as $\sigma \to 0$ the equilibrium converges to the Dudey equilibrium, as firms split their sales among the periods. For $\sigma > 0$ there is paradoxically stronger competition, even though, competition gets relaxed. The reason lies in the continuation profit $1-\sigma < 1$. The Dudey equilibrium requires that firms are able to agree for equal probability of sale in the first period at monopoly price, however this does not translate to equal split among periods, as there is always the chance that the consumer tomorrow is captive of the sold-out rival. This effect gets mitigated as σ becomes large and $\sigma \to \frac{1}{2}$, as then the market is exogenously evenly split between them in each period, leading to an optimal strategy to maximize the price and increase profits.

Case (ii): The Dudey equilibrium, where the constrained firm can sell in the first period at the monopoly price, no longer exists. The reason lies in the fact, that even if both firms are active in the second period, they expect strictly positive profits of σ. This gives an incentive for the unconstrained firm to undercut the monopoly price and sell today and expect a positive profit tomorrow.

There does not exist an equilibrium, where the unconstrained firm pushes the constrained firm to sell first at a price lower than the monopoly price. The reason is that the unconstrained firm has a share of captives to which it can always sell, thus, would charge them the monopoly price. A constrained firm would then have an incentive to increase its price to the monopoly price as well. Therefore, the monopoly price 1 should always be in the support of the equilibrium strategies of both firms.

Following the intuition of Dudey, where the unconstrained firm prefers to charge higher prices and let the constrained firm sell out first, we look for an equilibrium, where the unconstrained firm puts a mass ω at the top of the price distribution. The ex-ante expected profits of both firms are given by

$$\pi^C(p) = [\sigma + (1-2\sigma)(1 - F^U(p))]p + [\sigma + (1-2\sigma)F^U(p)]\sigma$$
$$\pi^U(p) = [\sigma + (1-2\sigma)(1 - F^C(p))](p + \sigma) + [\sigma + (1-2\sigma)F^C(p)](1-\sigma)$$

The optimal mixing strategy of the constrained firm, keeping the unconstrained firm indifferent between charging the monopoly price and any price in the support $p \in [\frac{\sigma+(1-2\sigma)^2}{1-\sigma}, 1)$ is by mixing according to $F^C(p) = 1 - \frac{\sigma(1-p)}{(1-2\sigma)(p-(1-2\sigma))}$. The unconstrained firm mixes over the same interval according to $F^U(p) = \frac{p-(1-2\sigma)}{p-\sigma} - \frac{\sigma(1-p)}{(1-2\sigma)(p-\sigma)}$ with a mass point $\omega = \frac{1-3\sigma}{1-\sigma}$ at the top. For $\sigma \in [0, \frac{1}{3}]$ this equilibrium retains some main characteristics from the perfect competition case. Namely, the unconstrained firm charges higher prices expecting to be a monopolist in the second period, while the constrained firm charges lower prices in order to sell first and avoid competition later. However, this is an equilibrium as long as $\sigma \leq \frac{1}{3}$.

For $\sigma \in (\frac{1}{3}, \frac{1}{2})$ the unconstrained firm prefers to compete for demand in both periods, as competition is relatively weak and the continuation profit is relatively high. In this case, the constrained firm prefers to aim and serve its captives at the maximum price, therefore it is the one setting a mass at the top. The unconstrained firm mixes according to $F^U(p) = 1 - \frac{\sigma(1-p)}{(1-2\sigma)(p-\sigma)}$ over the support $p \in [2\sigma, 1)$, while the constrained firm mixes according to $F^C(p) = \frac{(1-\sigma)(p-2\sigma)}{(1-2\sigma)(p-(1-2\sigma))}$ with a mass $\omega = \frac{3\sigma-1}{2\sigma}$ at the top. Interestingly, here as well, as $\sigma \to \frac{1}{2}$ the lower bound of the distribution converges to the monopoly price reflecting the decreasing competition and the convergence to even split of the market like in the case with two constrained firms.

Case (iii) with two unconstrained firms is equivalent to repeated imperfect Bertrand competition, where both firms mix over the interval $[\frac{\sigma}{1-\sigma}, 1]$ in each period according to $F(p) = 1 - \frac{\sigma(1-p)}{(1-2\sigma)p}$ and expect a profit of 2σ.

Incomplete Information about Capacities Let $\pi(\theta)$ denote the equilibrium continuation profit (of both firms) when the firm who sold is believed to be constrained with probability θ and let $\tilde{\pi}(p) := \pi(\theta(p))$. In particular $\pi(\theta) = \sigma + (1-2\sigma)\theta$ is affine in θ.

Proposition 1. *There exists some $\bar{\sigma} \in (0, 1/2)$ such that*

- *If $\sigma < \bar{\sigma}$, constrained firms randomize in $(\underline{p}, 1)$, and unconstrained firms randomize in $(\underline{p}, \bar{p})$, and $\theta(p)$ is increasing and convex in $(\underline{p}, \bar{p})$ and $\theta(\bar{p}) = 1$.*

- *If $\sigma > \bar{\sigma}$, both types of firms randomize in $(\underline{p}, 1)$, and $\theta(p)$ is increasing and convex in the whole support, with $\theta(\underline{p}) \in (\alpha, 1)$.*

Proof. The presence of captives implies that the upper bound of the price distribution must necessarily be 1 and that there are no mass points. To see why consider any upper bound lower than 1 and notice that the constrained type must be indifferent between selling now at that price or not (for otherwise, it would undercut). However, in this case it can profitably deviate to a higher price that has the same continuation profit in case it does not sell but a higher profit now. Captives makes this deviation strictly profitable. The same argument shows that no mass point can exist in the interior of the interval. To see that no mass point can exist at the upper bound, notice that captives imply that the continuation profit is bounded above by 1 and so constrained firms would prefer to undercut the mass point than expose themselves to losing to a rival with the same price.

It follows that there are only two types of equilibria, determined by some cutoff value $\bar{\sigma}$ (see below). First, if $\sigma > \bar{\sigma}$, the equilibrium involves both types of firms randomizing in an interval $(\underline{p}, 1]$, with $\underline{p}$ being the lowest price that a constrained firm is willing to charge. Namely,

$$(1-\sigma)\underline{p} + \sigma\pi(\alpha) = \sigma + (1-\sigma)\pi(\alpha).$$

In words, the constrained firm must be indifferent between charging $\underline{p}$ and selling to any consumer that is not captive with its rival now than charging the monopoly price, selling to its own captives. In both cases, the continuation profit in case of not selling is the same because not selling at the highest or the lowest price is uncorrelated with the type of the opponent. This yields,

$$\underline{p} = \frac{(1-2\sigma)\pi(\alpha)+\sigma}{1-\sigma}.$$

The same argument used as in the Proof of Proposition 4 in the case of homogeneous consumers shows that the continuation profit following any price in the support p is given by

$$\tilde{\pi}(p) = -\frac{p}{W_{-1}(-cp)},$$

where c is pinned down by the boundary condition $\tilde{\pi}(1) = \bar{\pi}$. In particular, $c(\bar{\pi}) = \frac{1}{\bar{\pi}\exp(\frac{1}{\bar{\pi}})}$, where we make explicit the dependence of c on the specific continuation profit at the upper bound of the price distribution. Finally, we can use the isoprofit condition of unconstrained firms to obtain

$$\tilde{\pi}(\underline{p})(1-\sigma) = \tilde{\pi}(1)\sigma = \bar{\pi}\sigma.$$

Combining these two yields the following equation:

$$1 + \frac{1-2\sigma}{\sigma}\pi(\alpha) = -\bar{\pi}W_{-1}\left(-c(\bar{\pi})\frac{(1-2\sigma)\pi(\alpha)+\sigma}{1-\sigma}\right).$$

This equation has a solution in $\bar{\pi} \in (\pi(\alpha), \pi(1))$ for every $\sigma \geq \bar{\sigma}$, where $\bar{\sigma}$ is implicitly defined as

$$1 + \frac{1-2\bar{\sigma}}{\bar{\sigma}}\pi(\alpha) = -\pi(1)W_{-1}\left(-c(\pi(1))\frac{(1-2\bar{\sigma})\pi(\alpha)+\bar{\sigma}}{1-\bar{\sigma}}\right).$$

Numerical results show that $\bar{\sigma}$ approaches 0 as $\alpha \to 0$ and $\bar{\sigma}$ approaches 1/2 as $\alpha \to 1$.

If $\sigma < \bar{\sigma}$, unconstrained firms cannot be lured to charge $p = 1$ no matter how high the continuation profits. In this case, the equilibrium requires that $\theta(p) = 1$ for $p \geq \bar{p}$, for some $\pi(1) < \bar{p} < 1$. Constrained firms randomize in $(\bar{p}, 1)$ trading off higher selling probability against a higher markup. This yields the following randomization condition:

$$pQ(p) + \int_{\underline{p}}^{\bar{p}} -Q'(y)\pi(y)dy + \pi(1)(Q(\bar{p}) - Q(p)) = \sigma + \int_{\underline{p}}^{\bar{p}} -Q'(y)\pi(y)dy + \pi(1)(Q(\bar{p}) - \sigma).$$

Notice that the continuation profit is flat for $p > \bar{p}$. Hence:

$$pQ(p) = \sigma + \pi(1)(Q(p) - \sigma)$$

This equation can be readily solved to obtain

$$Q(p) = \frac{\sigma(1 - \pi(1))}{p - \pi(1)}.$$

As before, in $(\underline{p}, \bar{p})$ both types of firms randomize. Their continuation profits are given by:

$$\tilde{\pi}(p) = \frac{-p}{W_{-1}(-cp)}.$$

We can solve for $c(\bar{p})$ using the fact that $\tilde{\pi}(\bar{p}) = \pi(1)$, resulting in

$$c(\bar{p}) = \frac{1}{\exp\left(\frac{\bar{p}}{\pi(1)}\right)\pi(1)}.$$

Finally, the equation that determines $\bar{p}$ is analogous as the case discussed before. Namely,

$$\tilde{\pi}(\underline{p})Q(\underline{p}) = \frac{-\underline{p}}{W_{-1}(-c(\bar{p})\underline{p})}(1 - \sigma) = Q(\bar{p})\pi(1).$$

E Additional Graphs

In section 4.1 of the main paper we derived a continuum of semi-separating equilibria, which differed in the interval of prices $[\alpha, \bar{p}]$ over which firms randomized their strategies. In particular, two semi-separating equilibria differ in the upper bound $\bar{p}$. As stated in Proposition 5 and illustrated by Figure 2 in the main paper, a higher $\bar{p}$ is associated with lower expected profits, as there is lower probability a constrained firm sells in the first period. In addition, Figure E.1 displays the probability of a constrained firm selling in the first period (blue, dashed) and the ex ante expected profit (red, solid) for different α. The graph shows that when the probability a constrained firm is in the market is low (first graph on the left for $\alpha = 0.2$), the odds ratio decreases quickly with $\bar{p}$. When the probability of a firm being constrained increases (the next three graphs), the odds ratio decreases at a lower pace, and profits increase as a result of the increased market power of firms. Nevertheless, profits decrease in $\bar{p}$ with the percentage difference between the pooling equilibrium and the equilibrium with $\bar{p} = 1$ decreasing as well from 16% to 1.6%.

Figure E.2 complements Figure 3 from the main paper by depicting the price distributions in the first (blue, dashed) and second (red, solid) period when α and $\bar{p}$ have different values.

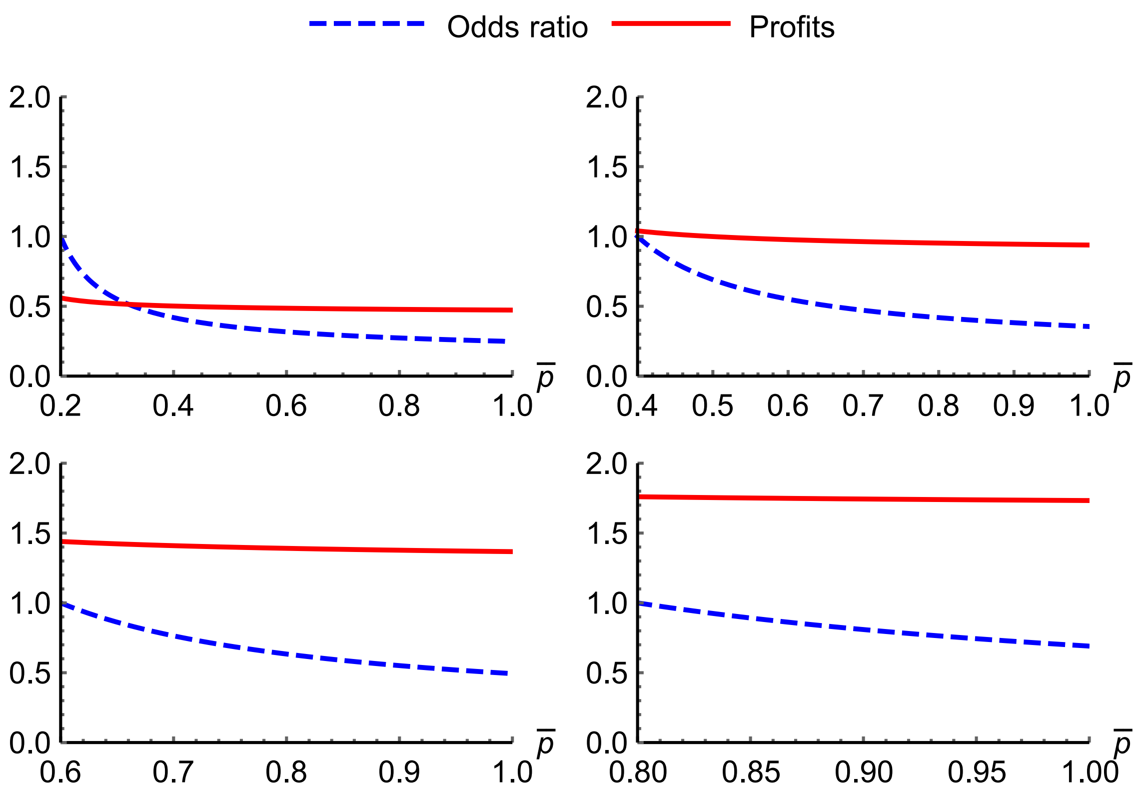

Figure E.1: This plot establishes the ex ante expected profit (red, solid) and the relative likelihood of a constrained firm selling in the first period (blue, dashed) in different equilibria (as a function of $\bar{p}$). Calibrated for (from left to right) $\alpha = \{0.2, 0.4, 0.6, 0.8\}$.

The Figure strengthens our result and shows that for different combinations of α and $\bar{p}$ the second-period price distribution is always wider, as firms mix over larger set of prices. The price distributions intersect only once, as the second-period prices are more dispersed due to the remaining uncertainty of the presence of a rival in the respective period.

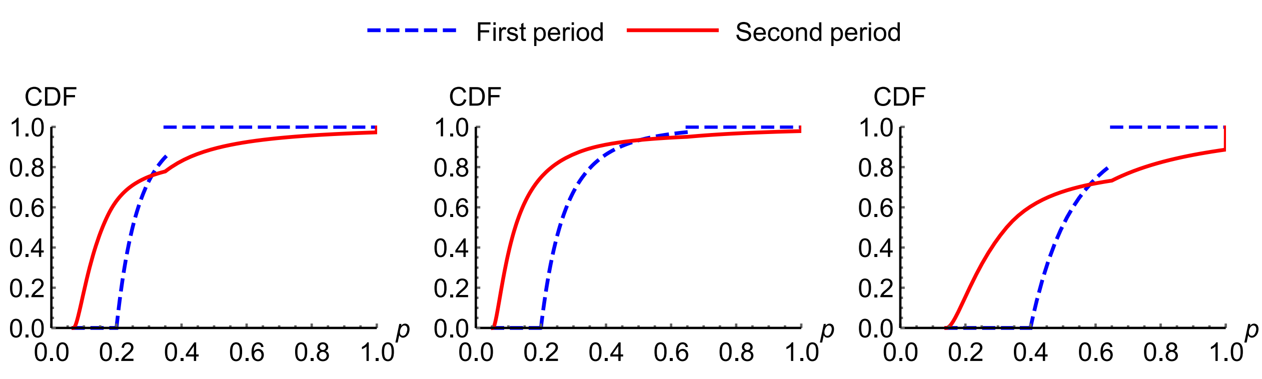

Figure E.2: First- (blue, dashed) and second-period (red, solid) price distributions. Calibrated for (from left to right) $\{(\alpha = 0.2, \overline{p} = 0.35), (\alpha = 0.2, \overline{p} = 0.65), (\alpha = 0.4, \overline{p} = 0.65)\},$.

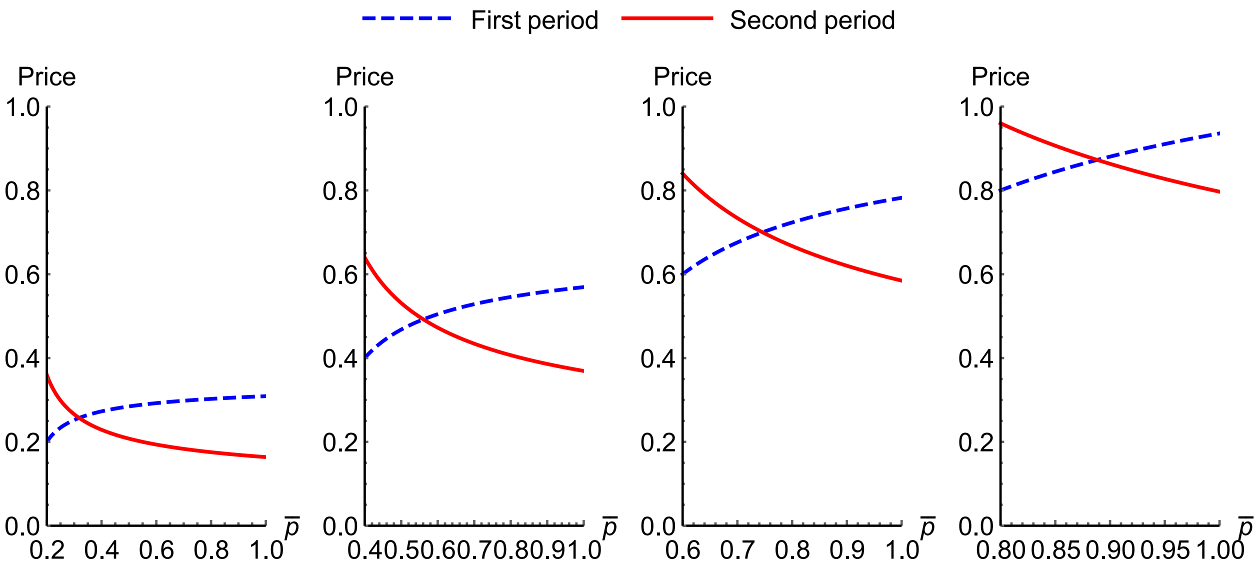

Figure E.3: This plot shows the ex ante expected first- (blue, dashed) and second-period (red, solid) transaction prices in different equilibria (as a function of $\bar{p}$). Calibrated for (from left to right) $\alpha = \{0.2, 0.4, 0.6, 0.8\}$.

Finally, Figure E.3 expands Figure 5 from the main paper for different α. It shows the expected prices in the first and second period for different semi-separating equilibria as a function of the upper bound of the distribution $\bar{p}$, for different values of α. In all semi-separating equilibria it holds that, as long as $\bar{p}$ is closer to α the second-period price is higher than the first-period price. When $\bar{p}$ becomes large and approaches 1, the probability that a constrained firm sells in the first period decreases (as depicted by Figure E.1 above), which results in lower posteriors and lower prices in the second period. Naturally, as α increases, so do the prices in both periods.