APPENDIX

(FOR ONLINE PUBLICATION ONLY)

In this appendix we formally define a belief refinement for Perfect Bayesian Equilibria (PBE) of our signaling game where two senders potentially have information about the same state of the world; the refinement is a natural extension of D1 refinement (Cho and Kreps, 1987) for the standard signaling game with single sender and single receiver. We then show that the beliefs underlying the PBE constructed in the proof of Propositions 1 and 3 satisfy this refinement.

Belief Refinement.

We have a signaling game with two senders (the two buyers) and one receiver (the seller) and both buyers may signal some information about the true asset quality to the uninformed seller. We outline two principles that must be satisfied by out-of-equilibrium beliefs.

For any PBE and realized price offer p_i^*, $i = 1, 2$, made by buyer i that is on the equilibrium path, one can use buyer $i's$ equilibrium strategy to derive the conditional probability $\mu_i(p_i^*)$ that asset quality is high. This conditional probability is based only on the price offered by buyer i. Now, consider a *unilateral* deviation by buyer j to an out of equilibrium price p, while the non-deviating buyer $i \neq j$ offers a price p_i^*. The first principle is that the *uninformed* seller should rely entirely on the information revealed by the non-deviating buyer's bid if her bid fully reveals quality for sure.

(P1) If $\mu_i(p_i^*) \in \{0, 1\}$ then regardless of the bid of the other buyer, the updated belief of the uninformed seller should be equal to $\mu_i(p_i^*)$.

Principle (P1) reflects the idea behind some other belief refinements in the literature such as Unprejudiced Beliefs due to Bagwell and Ramey (1991) that a single deviation is more likely than multiple deviations. The next principle specifies restrictions on beliefs when uncertainty about product quality remains after observing the non-deviating player's action.

Let $M_j(p)$ be the set of updated beliefs (updated probability of high quality) based on the out-of-equilibrium price p offered by player j that are consistent with an adaptation of the D1 criterion outlined below.

(P2) If $0 < \mu_i(p_i^*) < 1$ i.e., the non-deviating buyer $i's$ price does not reveal asset quality perfectly, then the out-of-equilibrium belief of the uninformed seller after observing price offers (p_i^*, p) can only assign probability $\mu(p_i^*, p)$ to high quality if

$$\mu(p_i^*, p) = \lambda\mu_i(p_i^*) + (1-\lambda)\mu_j \text{ for some } \mu_j \in M_j(p) \text{ and some } \lambda \in [0,1].$$

The refinement permits any out-of-equilibrium belief that is some weighted average of the beliefs derived separately from the non-deviating and the deviating player's actions. For instance, suppose that p_i^* is a pooling price that is offered in equilibrium with probability one by uninformed and H types; then $\mu_i(p_i^*) = \frac{\alpha}{\alpha\beta+(1-\beta)}$. If, further, the adapted D1 criterion assigns probability one to the deviating player being of type H, implying $M_j(p) = \{1\}$, then only out-of-equilibrium beliefs $\mu(p_i^*, p) \in \left[\frac{\alpha}{\alpha\beta+(1-\beta)}, 1\right]$ are consistent with principle (P2) of our refinement. On the other hand, if the adapted D1 criterion does not yield any restriction based on the deviating player's action i.e., $M_j(p)$ is the $[0,1]$ interval, then our refinement imposes no restriction on beliefs.

We now outline how we apply the D1 criterion to our setting under principle (P2). Consider a unilateral deviation to an out of equilibrium bid p by buyer j. To apply the D1 criterion, we need to determine the undominated responses of the seller for which this deviation is gainful (for each type of buyer j). As buyer j deviates before observing the bid of the non-deviating buyer $i \neq j$, we need to consider all possible equilibrium bids that may simultaneously come from buyer i and the seller's undominated response to each pair of bids that he might then observe. We have two reasonable restrictions here. When the equilibrium bid of buyer i is strictly higher than p, the seller sells to the deviating buyer j with probability zero. Further, if the equilibrium bid of buyer i reveals product quality fully (for sure), the uninformed seller's response is simply her optimal action based on the (deterministic) revealed quality. We can then determine the profiles of undominated responses by the uninformed seller (one for each possible equilibrium bid of the non-deviating buyer) for which each type of deviating buyer j earns an expected pay-off that is higher than her equilibrium payoff and compare the sets of such profiles for the various types of buyer j. If the set of such profiles of responses for one type of buyer j is a strict subset of that for another type, then the adapted D1 criterion assigns probability zero to the first type.

The equilibria constructed in the proof of Proposition 1 satisfy this refinement.

Consider the full pooling equilibrium constructed in the proof of part (a) of Proposition 1. As this is a symmetric full pooling equilibrium, the equilibrium action of the non-deviating buyer is uninformative and so our refinement suggests the uninformed seller use the D1 criterion to decide which buyer type of the deviating player has the strongest incentive to deviate to such a bid. We show that the H type buyer can gain from such an upward deviation for a larger set of undominated responses from the uninformed seller and therefore, the D1 criterion supports the specified belief. In particular, we check that the out-of-equilibrium belief (7) in the proof of Proposition 1 satisfies our belief refinement. Consider a unilateral deviation by buyer j to $p \in (V_L, c_H)$. As there is full pooling at price V_L, $\mu_i(V_L) = \alpha \in (0,1)$ so that only Principle (P2) can be applied. As $\mu_i(V_L) \in (0,1)$, every probability of selling is an undominated action of the uninformed seller after observing bids (p, V_L). Let $q_H(p), q_U(p)$ be the probabilities of selling by the uninformed seller that make the H and U type buyers indifferent between offering $p^* = V_L$ and deviating to p, i.e.,

$$q_H(p) = \frac{1}{1-\sigma}\left(\frac{S_H}{V_H - p}\right) = \frac{1}{2}\frac{V_H - V_L}{V_H - p}$$

$$q_U(p) = \frac{1}{1-\sigma}\left(\frac{S_U - \sigma(1-\alpha)(V_L - p)}{\alpha V_H + (1-\alpha)V_L - p}\right).$$

We show that $q_H(p) < q_U(p)$ for $p \in (V_L, c_H)$ so that using the D1 criterion, $M_j(p) = \{1\}$ and therefore, the specified belief $\mu(p^*, p) = 1$ meets principle (P2) of the refinement criterion R if we choose $\lambda = 0$. Note that $q_H(p) < q_U(p)$ if, and only if,

$$\frac{\alpha V_H + (1-\alpha)V_L - p}{V_H - p} < \alpha - \frac{2\sigma(1-\alpha)}{(1-\sigma)}\left[\frac{V_L - p}{V_H - V_L}\right].$$

As the left-hand side is strictly decreasing and the right-hand side is strictly increasing in p, the inequality holds for all $p \in (V_L, c_H)$ if, and only if, it holds weakly at $p = V_L$, which is true.

Next, consider the partial pooling equilibrium outlined in the proof of part (b) of Proposition 1. We can verify that the out-of-equilibrium belief restriction (12) satisfies our refinement criterion. Suppose the uninformed seller observes a unilateral deviation by buyer j to a price offer $p \in (p^*, c_H)$. If the non-deviating buyer i offers V_L, then as $\mu_i(V_L) = 0$ principle (P1) of our refinement implies $\mu(p, V_L) = 0$ and the only undominated action of the uninformed seller after observing pair of prices (p, V_L) is to sell with probability one. Now, suppose the non-deviating buyer i offers p^*, then $\mu_i(p_i^*) = \frac{\alpha}{\alpha\beta + (1-\beta)} \in (0,1)$ so that only Principle (P2) applies. For any pair of prices (p, p^*), as $\mu_i(p^*) \in (0,1)$, every probability of selling is an undominated action of the uninformed seller. We will now argue that an H type deviating buyer j has the strongest incentive to deviate to p using the D1 criterion so that $M_j(p) = \{1\}$ and therefore, the specified belief $\mu(p^*, p) = 1$ meets principle (P2) of the refinement criterion (by choosing $\lambda = 0$). Note that only an H type buyer has an incentive to deviate to $p > \alpha V_H + (1-\alpha)V_L$ and so $M_j(p) = \{1\}$ for such p. So, consider $p \in (p^*, \min\{\alpha V_H + (1-\alpha)V_L, c_H\})$. Let $q_\tau(p), \tau = H, U$ be the probability with which an uninformed seller sells after observing a bidder bidding p and the other bidding p^* such that a type τ buyer j is indifferent between deviating to bid p and not (where it is understood that in case the other buyer i offers V_L, the uninformed seller will sell for sure). Check that

$$q_H(p) = \frac{1}{1-\sigma}\left(\frac{S_H}{V_H - p}\right) = \frac{1}{2}\frac{V_H - p^*}{V_H - p},$$

$$q_U(p) = \frac{1}{2}\frac{(1-\alpha)(1+\beta)(V_L - p^*) + \alpha(1-\sigma)(V_H - p^*) - 2(1-\alpha)\beta(V_L - p)}{(1-\beta)(1-\alpha)(V_L - p) + \alpha(1-\sigma)(V_H - p)}.$$

It is sufficient to show that for all $p \in (p^*, \min\{\alpha V_H + (1-\alpha)V_L, c_H\})$, $q_H(p) <$

$q_U(p)$ which reduces to

$$\frac{V_H - p^*}{V_H - p} < \frac{(V_H - p^*) + \frac{(1-\alpha)(1-\beta)(V_L - p^*) + 2(1-\alpha)\beta(p-p^*)}{\alpha(1-\sigma)}}{(V_H - p) + \frac{(1-\beta)(1-\alpha)(V_L - p)}{\alpha(1-\sigma)}}.$$

which always holds.[8]

The equilibrium constructed in the proof of Proposition 3 satisfies this refinement.

Consider the partial pooling equilibrium constructed in the proof of Proposition 3. We show that the restriction (22) on out-of-equilibrium belief of the uninformed seller in the proof of Proposition 3 satisfies our refinement. Consider a unilateral deviation to price $p \in (V_L, c_H)$ by buyer j. The first restriction $\mu(p, p^*) = 1$, if $p^* \in [c_H, \overline{p}_H]$ follows directly from principle (P1) of the refinement. If the non-deviating buyer i offers $p^* = V_L$, as $\mu_i(V_L) = \frac{\alpha(1-\sigma)}{(1-\sigma)+\sigma(1-\alpha)} \in (0,1)$ principle (P1) does not apply but principle (P2) does and we can derive restrictions on beliefs based on the deviating buyer's incentives in the spirit of the D1 criterion. Note that at price (p, V_L), as $\mu_i(V_L) \in (0,1)$, every probability of selling is an undominated action of the uninformed seller. On other hand, for $p^* \in [c_H, \overline{p}_H]$, as $p < p^*$ the uninformed seller will never sell to the deviating buyer (it is a dominated action). We will now argue that U type of the deviating buyer j has the strongest incentive to deviate to p using the D1 criterion so that $M_j(p) = \{\alpha\}$ and the specified belief $\mu(p, V_L) = \alpha$ meets principle (P2) of the refinement criterion (by choosing $\lambda = 0$). It is clear that the L type buyer never has an incentive to deviate to $p > V_L$. When a H type buyer deviates to p, he cannot buy if either the rival buyer is informed or the seller is informed (as $p < c_H$). Let $q_\tau(p)$ be the probability of buying from an uninformed seller at price p when the rival buyer offers V_L that makes the

[8] This inequality can be rewritten as $\frac{V_H - p^*}{V_H - p} < \frac{V_H - p^* + x}{V_H - p + y}$ for appropriately chosen x and y; this holds if $(V_H - p^*)y < (V_H - p)x$ which reduces to

$$-(1-\beta)(V_H - V_L)(p - p^*) < 2\beta(V_H - p)(p - p^*)$$

and this always holds as the left hand side is negative, while the right hand side is positive.

τ type buyer indifferent between deviating and not deviating to p. Then,

$$\frac{1}{1-\beta}\frac{V_H - c_H}{V_H - p} = q_H(p)$$

$$q_U(p) = \frac{(1-\alpha)\sigma(p - V_L)}{(1-\sigma)\{(1-\beta)\alpha(V_H - p) + (1-\alpha)(V_L - p)\}}.$$

It follows that $q_H(p) \geq q_U(p)$ iff

$$(1-\beta)\alpha + (1-\alpha)\frac{V_L - p}{V_H - p} \geq \frac{(1-\alpha)\sigma(p - V_L)}{V_H - c_H}$$

and as the right hand expression is increasing in p and the left hand expression is decreasing in p the inequality holds for all $p \in (V_L, c_H)$ iff it holds at $p = V_L$, which is obviously the case. This concludes the proof.